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Thermalization of topological order

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Joint work with David Pérez-García (UCM/ICMAT), Antonio Pérez-Hernández (UNED).

Outline

Motivation: self-correcting quantum memories

Thermalization: mathematical overview

Quantum Double Models

The PEPS parent Hamiltonian

Motivation: self-correcting quantum memories

Error correcting codes

Example: (N,1) repetition code

encoding: $0 \mapsto 0 \dots 0, 1 \mapsto 1 \dots 1$
decoding: majority vote.

We can correctly recover the encoded bit if we have at most $\lfloor N/2 \rfloor - 1$ errors.

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We can think of a classical spin model as a repetition code:

Ising model
We encode each bit into a spin: $0 \mapsto +1, \quad 1 \mapsto -1$ where $+1$ and -1 are two distinguished spin configuration of a particle/atom.

The energy of a collection of spins $\vec{s} = \{s_1, \dots, s_N\}$, $s_i = \pm 1$, is given by

$$H(\vec{s}) = -J \sum_{i \sim j} s_i s_j, \quad J > 0,$$

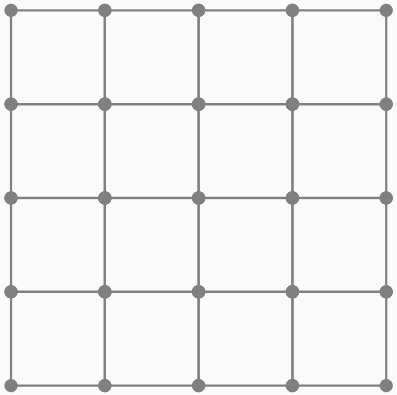
where $i \sim j$ means that spin i is a "neighbor" of spin j.

Codewords: configurations with minimal energy.

1D vs 2D

1D: $\mathbb{Z}_N$

2D: $\mathbb{Z}_N^2$



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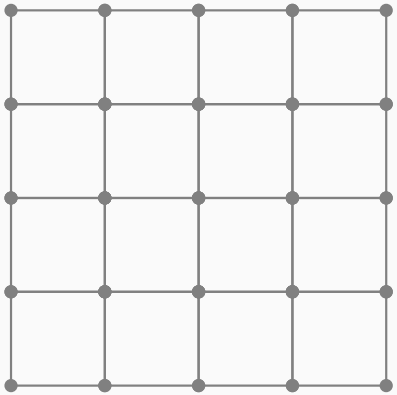

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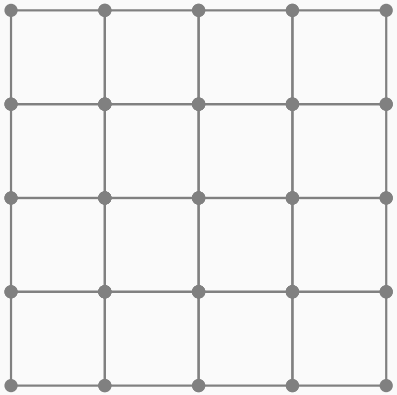

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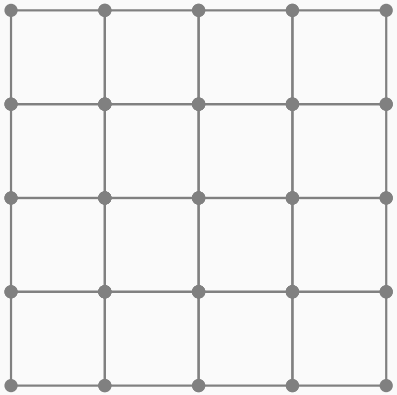

point-like elementary excitations

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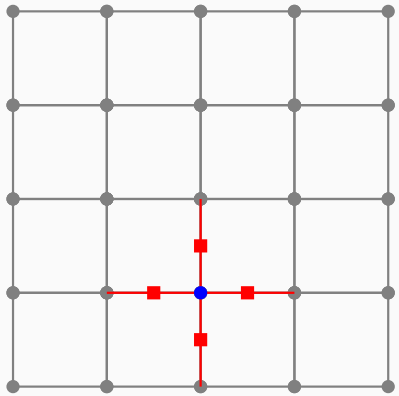

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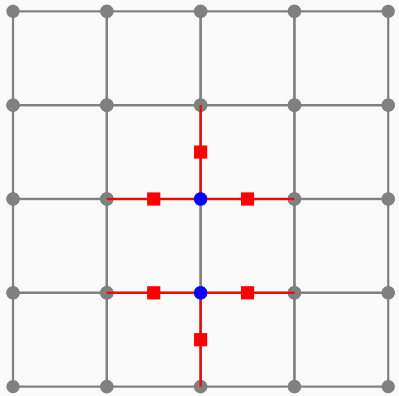

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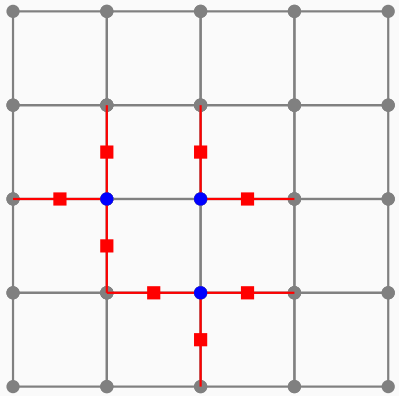

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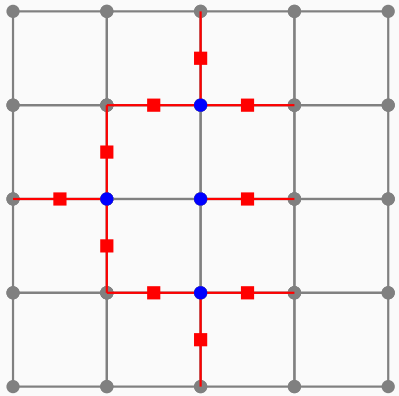

line-like elementary excitations

Energy barrier

Assuming our noise model can flip 1 bit at a time:

1. In 1D, we can go from a codeword to a different one, with constant energy;
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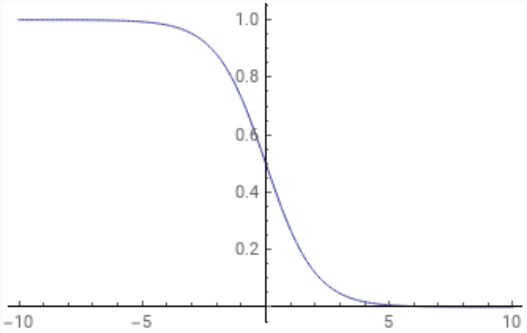
Thermalization of classical memories

Glauber's dynamics at temperature $T = 1/\beta$

1. Choose a site at random: x
2. Let ΔE be the energy difference if we flip spin at x.
   We accept the change with probability

$$p(\Delta E) = \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}}$$



3. Repeat.

This defines an ergodic Markov process, whose stationary state is Boltzmann distribution:

$$p(\vec{s}\,) \sim \exp(-\beta H(\vec{s}\,))$$

1D vs 2D: thermalization

Mixing time
How long does it take for Glauber's dynamics for the Ising model to reach (close) to equilibrium, as a function of system size N?

1. In 1D, polynomial, for every T;
2. In 2D, for T sufficiently small, takes exponential time!

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1. In 1D, polynomial, for every T;
2. In 2D, for T sufficiently small, takes exponential time!

In the latter case, we say that the memory is **self-correcting**.

Note: 1D Ising model has no critical temperature / phase transition, 2D Ising model does.

Quantum memories

Quantum error correcting codes

Quantum Hamiltonian H on N qudits $\mathcal{H} := (\mathbb{C}^d)^{\otimes N}$ with ground state space dimension k.

We can encode a vector in $\mathbb{C}^k$ into the ground state space

$$\mathbb{C}^k \ni |\phi\rangle \mapsto |\hat{\phi}\rangle \in \mathrm{groundspace}(H) \subset (\mathbb{C}^d)^{\otimes N}$$

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Topological quantum memories

Since topological ordered models have ground states which are *locally indistinguishable*, they seem to be a good fit for a quantum error correcting code.

Which topological ordered models are self-correcting at finite temperature?

Related question: existence of critical temperature for topological ordered models.

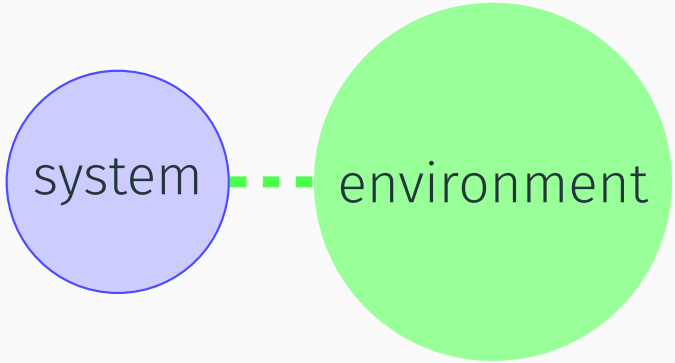
Thermalization: mathematical overview

How to model the effect of temperature?

$$T > 0$$

Thermalization

Thermal bath
Systems interacting with some (large) thermal bath at temperature $T > 0$.



Joint Hamiltonian:

$$H(\lambda) := H_{\text{sys}} \otimes \mathbb{1}_{\text{env}} + \mathbb{1}_{\text{sys}} \otimes H_{\text{env}} + \lambda \sum_{\alpha} S_\alpha \otimes B_\alpha \quad \lambda \geq 0$$

Initial state:

$$\rho_0 \otimes \sigma_\beta, \quad \sigma_\beta := \frac{1}{Z_\beta} \exp(-\beta H_{\text{env}}), \quad \beta = \frac{1}{T}$$

System evolution:

$$\rho_{\text{sys}}(t) = \text{Tr}_{\text{env}}[U_{\lambda,t}(\rho_0 \otimes \sigma_\beta)U_{\lambda,t}^\dagger], \quad U_{\lambda,t} = \exp(itH(\lambda))$$

Effective evolution

$\rho_0 \mapsto \rho_{\mathrm{sys}}(t)$ is linear, completely positive, trace preserving (CPTP) for each $t > 0$, but hard to describe!

Markovian / Lindbladian effective evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{eff}}(t) = \mathcal{L}(\rho_{\mathrm{eff}}(t)) \quad \forall t \geq 0, \qquad \rho_{\mathrm{eff}}(0) = \rho_0$$

- $\exp(t\mathcal{L})$ is a CPTP map for every $t \geq 0$
- $\rho_\beta = \frac{1}{Z_\beta} \exp\bigl(-\beta H_{\mathrm{sys}}\bigr)$ is invariant: $\mathcal{L}(\rho_\beta) = 0$

Davies semigroup

Davies' generator [Comm. Math. Phys. 1974]

$$\mathcal{L}(\rho) = -i\big[H_{\mathrm{sys}}, \rho\big] + \lambda^2 \mathcal{D}(\rho)$$

where

$$\mathcal{D}(\rho) = \sum_{\alpha} \sum_{\omega} \hat{g}_{\alpha}(\omega) \left(S_{\alpha}(\omega) \rho S_{\alpha}(\omega)^{\dagger} - \frac{1}{2} \{ \rho, S_{\alpha}(\omega)^{\dagger} S_{\alpha}(\omega) \} \right)$$

ω: Bohr frequencies of H_{sys} (differences of eigenvalues)
$\hat{g}_{\alpha}(\omega) \geq 0$: depend on autocorrelation function of the bath, satisfy *detailed balance* $\hat{g}_{\alpha}(-\omega) = e^{-\beta\omega} \hat{g}_{\alpha}(\omega)$

$$S_{\alpha} = \sum_{\omega} S_{\alpha}(\omega), \quad e^{itH_{\mathrm{sys}}} S_{\alpha} e^{-itH_{\mathrm{sys}}} = \sum_{\omega} e^{it\omega} S_{\alpha}(\omega) g$$

Properties of Davies semigroup

- $\mathcal{L}$ generates a CPTP semigroup;
- $\rho_\beta = \frac{1}{Z_\beta} \exp\left(-\beta H_{\text{sys}}\right)$ is invariant: $\mathcal{L}(\rho_\beta) = 0$;
- If $\{S_\alpha\}'_\alpha = \mathbb{C}\mathbb{1}$, then the semigroup is ergodic: ρ_β is the unique fixed point

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In many cases, $\rho_{\mathrm{eff}}(t)$ approximates $\rho_{\mathrm{sys}}(t)$ in the *weak-coupling limit*:

- Davies, 1974-1976: under certain assumptions on the thermal bath, for every $\tau > 0$

$$\lim_{\lambda \to 0} \sup_{0 \leq \lambda^2 t < \tau} \left\| \rho_{\mathrm{eff}}(t) - \rho_{\mathrm{sys}}(t) \right\|_1 = 0$$

- Merkli, 2020:

$$\sup_{t \geq 0} \left\| \rho_{\mathrm{eff}}(t) - \rho_{\mathrm{sys}}(t) \right\|_1 \leq C\lambda^2$$

(dependence of C on system size not clear)

Locality of Davies generator

H_{sys} is a quantum Hamiltonian on $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$, which we assume are arranged on a graph or lattice.

If we assume

- H_{sys} is a *local* commuting Hamiltonian

$$H_{\mathrm{sys}} = \sum_p H_p, \quad [H_p, H_{p'}] = 0 \quad \forall p, p'$$

- the thermal coupling is i.i.d. on each site: $S_{\alpha,i}$ acts on site i for each $i = 1, \ldots, N$

then $S_{\alpha,i}(\omega)$ is *local*:

$$\sum_\omega e^{it\omega} S_{\alpha,i}(\omega) = e^{itH_{\mathrm{sys}}} S_{\alpha,i} e^{-itH_{\mathrm{sys}}} = e^{itH_{N(i)}} S_{\alpha,i} e^{-itH_{N(i)}}$$

where ω runs over the Bohr frequencies of $H_{N(i)} = \sum_{p \ni i} H_p$.

Local Lindbladian

In this case, the Davies generator is also local

$$\mathcal{D} = \sum_i \mathcal{D}_i$$

$$\mathcal{D}_i(\rho) = \sum_{\alpha,\omega} \hat{g}_{\alpha,i}(\omega)\left(S_{\alpha,i}(\omega)\rho S_{\alpha,i}(\omega)^\dagger - \frac{1}{2}\{\rho, S_{\alpha,i}(\omega)S_{\alpha,i}(\omega)^\dagger\}\right)$$

Assumption 1: translation invariance

1. $\hat{g}_{\alpha,i} = \hat{g}_\alpha$
2. $S_{\alpha,i}(\omega)$ and $S_{\alpha,j}(\omega)$ are related by translation

Assumption 2: "local ergodicity":

$$\{S_{\alpha,i}\}'_\alpha = \mathbb{C}\mathbb{1} \quad \forall i$$

Relaxation time

Spectral gap $\mathrm{gap}(\mathcal{D})$**:** Difference between second largest and largest eigenvalue (in modulus).

The spectral gap controls how quickly the dissipation happens.

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Scaling?

- If $\mathrm{gap}(\mathcal{D}) = \Omega(1)$ as system size N grows, then the mixing time is polynomial in N and we have *no self-correction*.
- If $\mathrm{gap}(\mathcal{D}) = o(N)$ (for low enough T) then we might have *self-correction*.

Is 2D self-correction possible?

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Probably not.

State of the art

- The 2D Toric Code is not self-correcting: $\mathrm{gap}(\mathcal{D}) = \Omega(e^{-\beta c})$ uniform in N
  (Alicki-Fannes-Horodecki 2008).
- The 4D Toric Code is self-correcting: $\mathrm{gap}(\mathcal{D}) = \mathcal{O}(e^{-cN})$ for low T
  (Dennis, Kitev, Landahl, Preskill 2002, Alicki-Horodecki$^{\otimes 3}$ 2010)
- For Quantum Double models (Kitaev 1997) with G abelian $\mathrm{gap}(\mathcal{D}) = \Omega(e^{-\beta c})$ uniform in N
  (Kómár, Landon-Cardinal, Temme, 2016)
- Heuristic arguments based on energy barriers (Bravyi, Terhal 2009; Landon-Cardinal, Poulin 2013)

Our result

[arxiv:2107.01628, Forum of Mathematics, Sigma 2023]
For any quantum double model with group G, we have that for any $\beta < \infty$

$$\mathrm{gap}(\mathcal{D}) \geq \lambda(G, \beta) > 0$$

uniformly in N.

What's interesting about our result:

1. Covers non-abelian models (required for universal quantum computation by braiding anons)
2. We use tensor networks techniques!

What's not so nice:

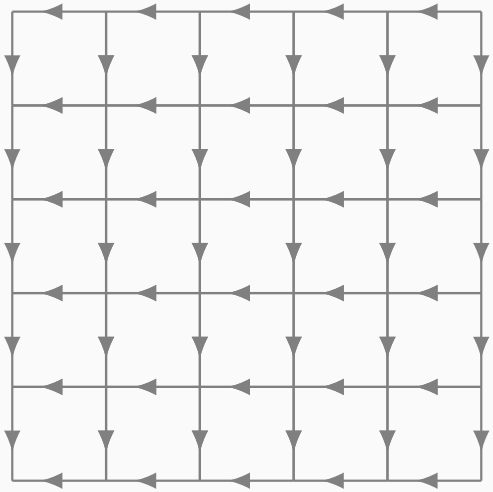
- Dependence on β is worse than previous results ...
- ...but we have ideas on how to improve it and obtain a similar bound.

Quantum Double Models

Quantum double model with group G

Quantum doubles

- $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$ with orientation



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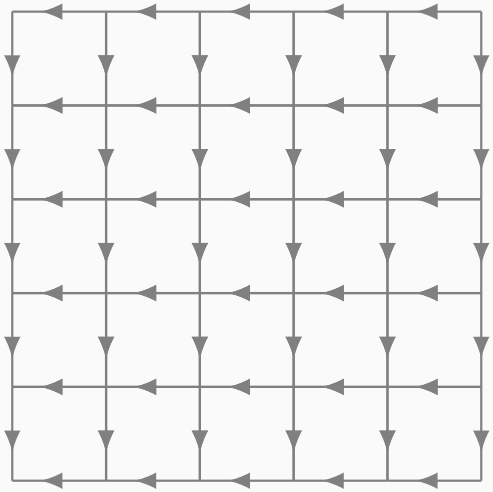

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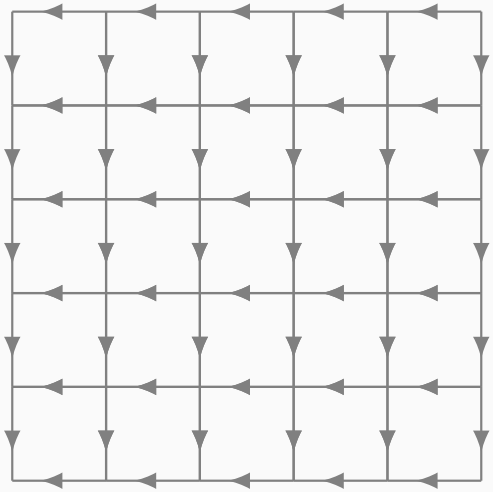
- for each edge $e \in \mathcal{E}$

$$\mathcal{H}_e = \ell_2(G)$$



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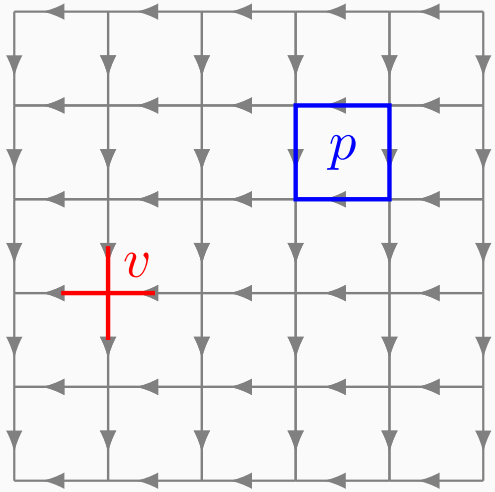
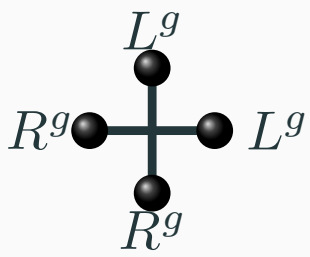
- Commuting Hamiltonian

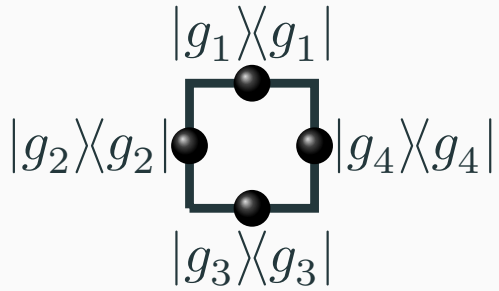
$$H_\Lambda = -\sum_{v \text{ vertex}} A_v \; - \sum_{p \text{ plaquette}} B_p$$



where $[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0$

The interaction terms

$$A_v = \frac{1}{|G|} \sum_{g \in G}$$ 

$$B_p = \frac{1}{|G|} \sum_{g_4^{-1} g_3^{-1} g_2 g_1 = 1}$$ 

where

$$L^g := \sum_{h \in G} |gh\rangle\langle h|\,, \qquad L^g\,|h\rangle = |gh\rangle$$
$$R^g := \sum_{h \in G} \left|hg^{-1}\right\rangle\langle h|\,, \qquad R^g\,|h\rangle = \left|hg^{-1}\right\rangle$$

are the **left and right regular representations** of G.

Some properties

1. **The ground state dimension is independent of N.**
   It is given by the number of flat G-connections or

$$\dim\left(\mathrm{Hom}(\mathbb{Z}^2, G)/\!\sim G\right)$$

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2. **Elementary excitations are localized particles.**
   They can classified by the irreps of $D(G)$ Drinfield's quantum double algebra of $\ell_2(G)$.
3. **They can support universal quantum computation.**
   If G is not abelian, braiding and exchanging elementary excitation is a non-abelian operation. This allows to encode arbitary quantum computations (topological quantum computing).

Detailed balance

GNS scalar product

$$\langle A, B\rangle_\beta = \operatorname{Tr} \rho_\beta A^\dagger B, \quad \rho_\beta = \frac{1}{Z_\beta} e^{-\beta H_\Lambda}$$

Detailed balance

$-\mathcal{D}$ is self-adjoint and positive semidefinite with respect to $\langle \cdot, \cdot \rangle_\beta$

$$\langle \mathcal{D}(A), B\rangle_\beta = \langle A, \mathcal{D}(B)\rangle_\beta, \quad \langle \mathcal{D}(A), A\rangle_\beta \leq 0.$$

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We can represent $-\mathcal{D}$ as a local Hamiltonian $\tilde{H}$ such that

$$\operatorname{gap}(\mathcal{D}) = \operatorname{gap}(\tilde{H})$$

Vectorization

We can identify $\mathcal{B}(\mathcal{H}_\Lambda)$ with $\mathcal{H}_\Lambda^2 = \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$ via

$$Q \mapsto |\iota(Q)\rangle = (Q\rho_\beta^{1/2} \otimes \mathbb{1}) |\Omega\rangle , \quad |\Omega\rangle = \sum_{\vec{g} \in G^\Lambda} |\vec{g}, \vec{g}\rangle$$

It is an isometry between $(\mathcal{B}(\mathcal{H}_\Lambda), \langle \cdot, \cdot \rangle_\beta)$ and $\mathcal{H}_\Lambda^2$.

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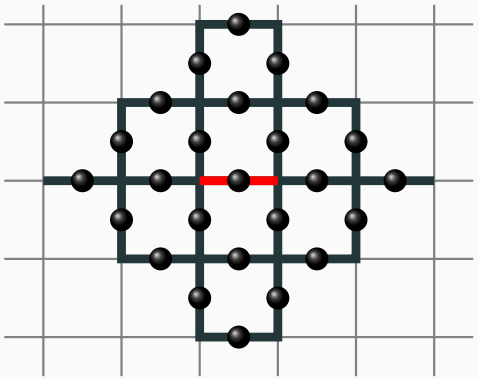
Vectorized Lindbladian

$$\tilde{H}_e |\iota(Q)\rangle := -|\iota(\mathcal{D}_e(Q))\rangle$$

$$\tilde{H} = \sum_e \tilde{H}_e, \quad \tilde{H}_e \geq 0$$

$\tilde{H}$ is a local, frustration-free Hamiltonian!



Support of $\widetilde{H}_e$

Ground states

We can characterize the local ground states subspaces of $\tilde{H}$:

$$\ker \sum_{e \in X} \mathcal{D}_e = \{S_{\alpha,e}(\omega) \mid e \in X\}' \subset \mathbb{1}_X \otimes \mathcal{B}(\mathcal{H}_{X^c})$$

$$\ker \sum_{e \in X} \tilde{H}_e \subset \{(Q\rho_\beta^{1/2} \otimes \mathbb{1})\,|\Omega\rangle \mid Q \in \mathcal{B}(\mathcal{H}_{X^c})\}$$

In particular, when $X = \Lambda$, the unique ground state of $\tilde{H}$ is the *thermofield double* $|\rho_\beta^{1/2}\rangle = |\iota(\mathbb{1})\rangle$, which is a local purification of ρ_β.

Warning!

- H_Λ: quantum double model Hamiltonian on $\mathcal{H}_\Lambda$, commuting, degenerate ground states;
- $\tilde{H}$: non-commuting Hamiltonian on $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$, unique ground state $|\rho_\beta^{1/2}\rangle$, $\mathrm{gap}(\tilde{H}) = \mathrm{gap}(\mathcal{D})$

Tensor networks tools

1. The thermofield double state $|\rho_\beta^{1/2}\rangle$ can be represented as a PEPS (Projected Entangled Pair State)
2. PEPS have a frustration-free Hamiltonian associated to them, called the parent Hamiltonian H^{parent}
3. There are conditions on the PEPS that imply that H^{parent} has a spectral gap.

Theorem
Let $\mathcal{X}$ be the family of rectangles having at most $n(\beta)$ plaquettes per row and column, for $n(\beta) \sim \exp(\beta)$. Then there exists a parent Hamiltonian with unique ground state $|\rho_\beta^{1/2}\rangle$

$$H^{parent} = \sum_{X \in \mathcal{X}} P_X^{\perp}$$

such that $\text{gap}(H^{parent}) \geq \kappa > 0$ uniformly in β and N.

We want to compare $\tilde{H}$ with H^{parent}!

Comparison

Lemma (Step 1: intermediate projection)
Let Π_X the projection on

$$\{(Q\rho_\beta^{1/2} \otimes \mathbb{1}) |\Omega\rangle \mid Q \in \mathcal{B}(\mathcal{H}_{X^c})\}$$

Then

$$\sum_{e \in X} \tilde{H}_e \geq \frac{C}{|X|} e^{-c\beta|X|} \Pi_X^\perp, \quad \textit{and} \quad \Pi_X^\perp \geq P_X^\perp.$$

Lemma (Step 2: coarse-graining)

$$\tilde{H} = \sum_{e \in \Lambda} \tilde{H}_e \geq \frac{1}{n(\beta)^4} \sum_{X \in \mathcal{X}} \left(\sum_{e \in X} \tilde{H}_e \right) \geq C \frac{e^{-c\beta n(\beta)^2}}{n(\beta)^4} H^{\textit{parent}}$$

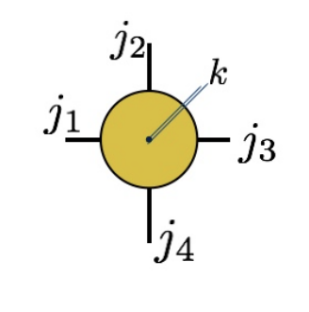
Conclusion

$$\mathrm{gap}(\mathcal{D}) = \mathrm{gap}(\tilde{H}) \geq C \frac{e^{-c\beta n(\beta)^2}}{n(\beta)^4} \kappa \quad \text{uniformly in } N$$

The PEPS parent Hamiltonian

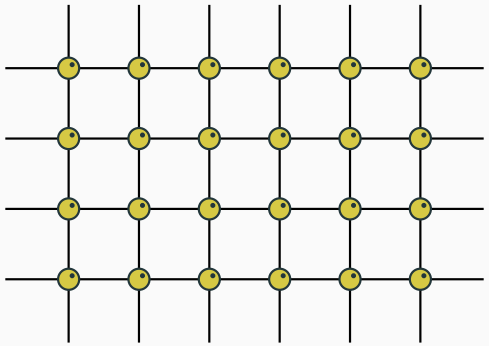
PEPS and Parent Hamiltonian

What is a PEPS? It is a family of states, defined by the contraction of tensors $V_e \in (\mathbb{C}^D)^{\otimes 4} \otimes \mathbb{C}^d$. D is called the *virtual* dimension, d the *physical* dimension.



For each region $\mathcal{R}$, contracting virtual indices in $\mathcal{R}$ gives a linear map from boundary virtual indices to physical indices:

$$V_{\mathcal{R}} : \mathcal{H}_{\partial\mathcal{R}} \longrightarrow \mathcal{H}_{\mathcal{R}}$$



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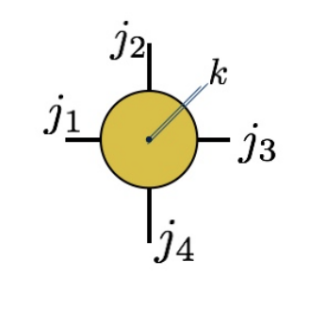

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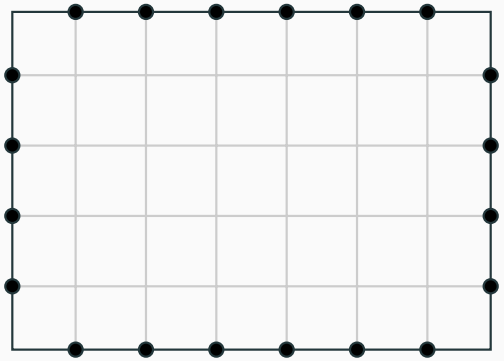

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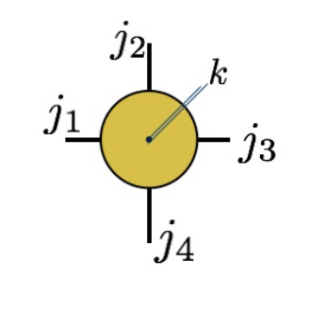

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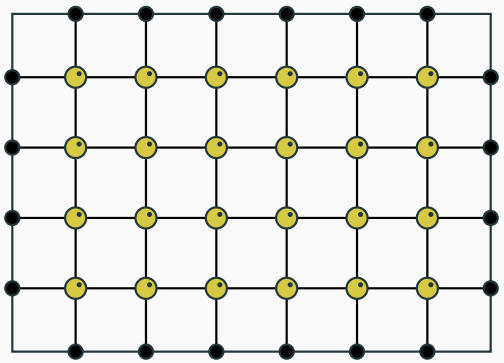
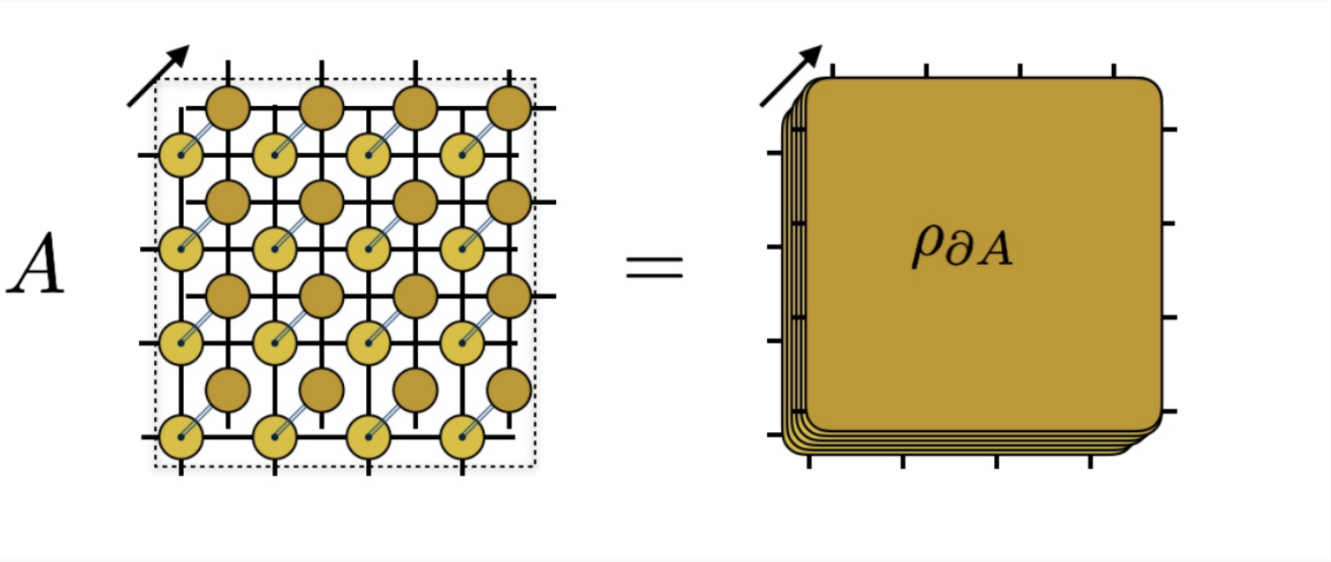


Parent Hamiltonian and approximate factorization

Parent Hamiltonian
A local f.f. Hamiltonian whose ground states in a region $\mathcal{R}$ are **exactly** $\operatorname{Ran} V_{\mathcal{R}}$, for a "good" class of regions $\mathcal{R}$ (e.g. all sufficiently large rectangles).

Boundary state
The boundary state of a region $\mathcal{R}$ is a (unormalized) density matrix on the boundary virtual indices, obtained by tracing out the physical indices.

$$\rho_{\partial\mathcal{R}} = V_{\mathcal{R}}^{\dagger} V_{\mathcal{R}} \in \mathcal{B}(\mathcal{H}_{\partial\mathcal{R}})$$



Approximate factorization

Definition (Approximate factorization)
$\rho_{\partial\mathcal{R}}$ is ϵ-approximately factorizable if there exists a product state

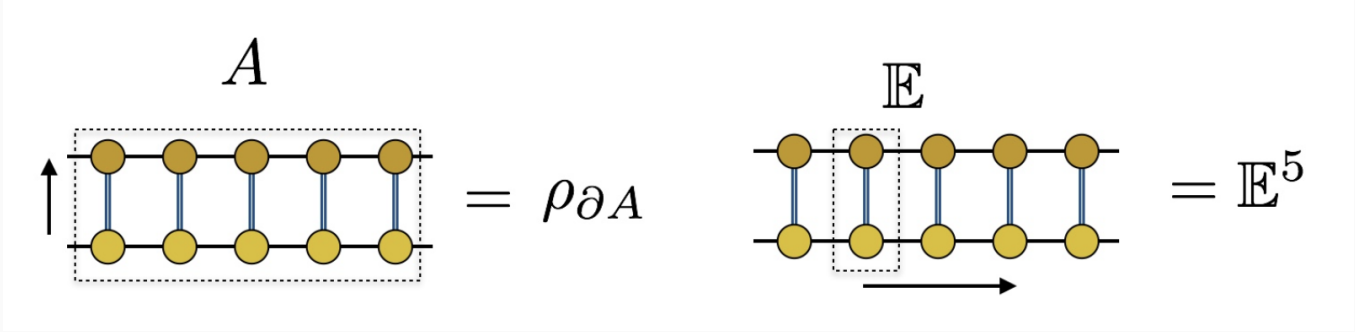
$$\sigma_{\partial\mathcal{R}} = \bigotimes_{u\in\partial\mathcal{R}} \sigma_x$$

such that

$$\|\rho_{\partial\mathcal{R}}^{1/2}\sigma_{\partial\mathcal{R}}^{-1}\rho_{\partial\mathcal{R}}^{1/2} - \mathbb{1}\| \leq \varepsilon$$

Kastoryano-L.Perez-Garcia arXiv:1709.07691
For an appropriately chosen family of rectangles $\{\mathcal{R}\}$, if $\rho_{\mathcal{R}}$ is $\epsilon(\mathcal{R})$-approximately factorizable, with $\epsilon(\mathcal{R})$ decaying exponentially in the diameter of $\mathcal{R}$, then the parent Hamiltonian has a positive spectral gap (independent of system size).

Approximate factorization – 1D



In 1D, the boundary state is the **Choi-Jamiołkowski** matrix of the **transfer operator**.

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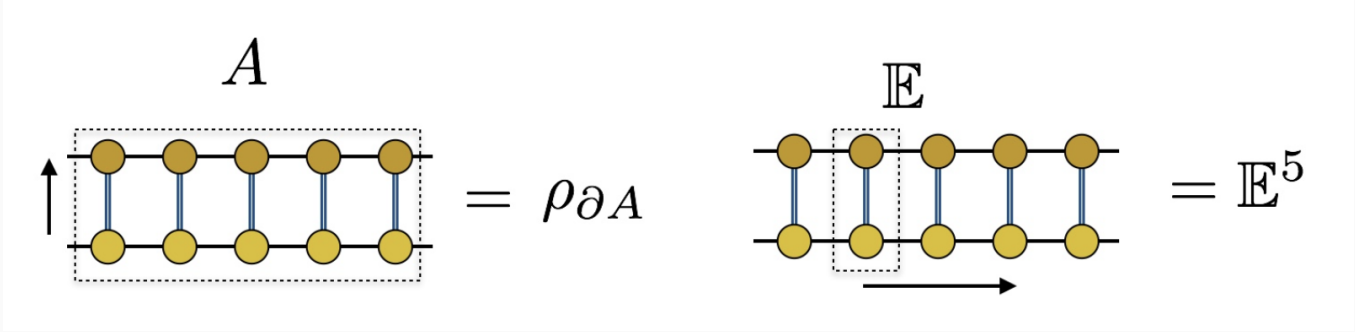


In 1D, the boundary state is the **Choi-Jamiołkowski** matrix of the **transfer operator**. In the tensor is *injective*, then $\mathbb{E}$ is primitive: $\mathbb{E}^n$ converges to a rank-one operator. This implies that

$$\rho_{\partial\mathcal{R}} \to \sigma = \sigma_1 \otimes \sigma_2$$

exponentially fast in $|\mathcal{R}|$. It implies approximate factorization since

$$\left\| \rho_{\partial\mathcal{R}}^{1/2} \sigma_{\partial\mathcal{R}}^{-1} \rho_{\partial\mathcal{R}}^{1/2} - \mathbb{1} \right\| \leq \sigma_{\min}^{-1} \| \rho_{\partial\mathcal{R}} - \sigma \|$$

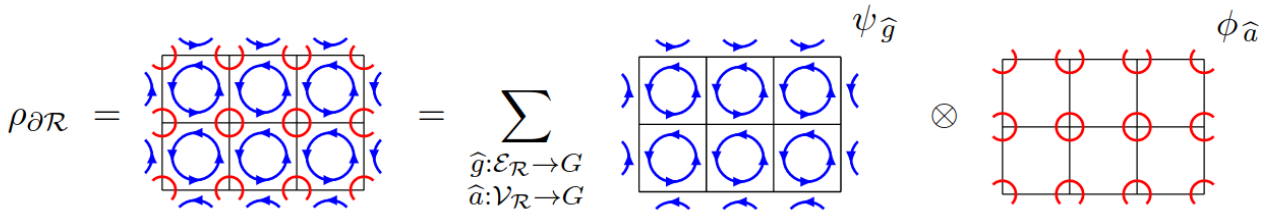
Locality of the boundary – quantum double

Using the explicit description of $|\rho_\beta^{1/2}\rangle$ as a PEPS, we can bound:

$$\|\rho_{\partial\mathcal{R}} - \kappa_{\partial\mathcal{R}} J_{\partial\mathcal{R}}\|_\infty \leq 3|G|^2 \left(\frac{\gamma_\beta}{1+\gamma_\beta}\right)^{|\mathcal{R}|}, \quad \gamma_\beta = \frac{e^\beta - 1}{|G|}$$

where $J_{\partial\mathcal{R}}$ is the projection on the support of $\rho_{\partial\mathcal{R}}$.

formalize. We are going to expand the expression for $\rho_{\partial\mathcal{R}}$



as a sum over maps

$$\widehat{g} : \mathcal{E}_\mathcal{R} \longrightarrow G \quad , \quad \widehat{a} : \mathcal{V}_\mathcal{R} \longrightarrow G$$

satisfying a certain compatibility condition. Let us explain the notation: at each

This implies approximate factorization, choosing $\sigma_{\partial\mathcal{R}} = \kappa_{\partial\mathcal{R}} J_{\partial\mathcal{R}}$

Outlook

- For all 2D quantum double models with group G

$$\mathrm{gap}(\mathcal{D}) \geq \lambda(G, \beta) > 0$$

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Thank you for your attention!

1D models

No-self correction in 1D

Self-correction is impossible in 1D even for **classical** systems:

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2. Kastoryano, Brandao 2016: spectral gap for 1D commuting quantum spin chains

log-Sobolev inequality for 1D models

Bardet, Capel, Gao, L. Pérez-García, Rouzé 2023

For a commuting 1D spin chain Hamiltonian, the associated ergodic Davies generator at any inverse temperature $\beta < \infty$ satisfies a log-Sobolev inequality with constant

$$\alpha(\mathcal{D}) \geq C \log^{-1}(N)$$

which implies that the mixing time is logarithmic in N.

This is a culmination of a long line of work:

1. Bardet, Capel, Gao, L., Pérez-García, Rouzé. arXiv:2112.00601
2. Bardet, Capel, Gao, L., Pérez-García, Rouzé. Phys. Rev. Lett. 130 (2023)
3. Bardet, Capel, L., Pérez-García, Rouzé. J. of Math. Phys. 62 (2021)
4. Capel, L., Pérez-García. J. of Phys. A 51.48 (2018)
5. Capel, L., Pérez-García. IEEE Trans. Inf. Th. 64.7 (2017)

SPT phases

Symmetries

If the system has a symmetry G, with unitary representation u_g such that

$$\left[H_{\text{sys}}, u_g\right] = 0 \quad \forall g \in G$$

then we could require the existence of a representation U_g on the environment such that

$$\left[H(\lambda), u_g \otimes U_g\right] = 0 \quad \forall g \in G$$

In this case, $\mathcal{L}$ satisfies a *covariance* condition:

Weak symmetry

$$\mathcal{L}(u_g \rho u_g^\dagger) = u_g \mathcal{L}(\rho) u_g^\dagger$$

Symmetries

$$\left[H(\lambda), u_g \otimes U_g\right] = 0 \quad \forall g \in G$$

A stronger requirement is that $U_g = \mathbb{1}$ (trivial representation), in which case

Strong symmetry

$$\mathrm{Tr}\left[u_g \mathcal{L}(\rho)\right] = 0, \quad \text{or equivalently that } \left\langle u_g \right\rangle_{\rho_{\mathrm{eff}}(t)} \text{ constant } \forall g \in G$$

A sufficient (and necessary, if $\mathcal{L}$ has a full rank invariant state) is that

$$\left[S^\alpha, u_g\right] = 0 \quad \forall \alpha, \forall g \in G$$

Strong symmetry vs. ergodicity

If $\mathcal{L}$ is strongly symmetric, then it **cannot** be ergodic!

$$\{S_\alpha\}'_\alpha \supseteq \{u_g \mid g \in G\}$$

If u_g is not irreducible, each irrep-sector is invariant, and

$$\rho_\beta = \sum_\gamma \rho_\beta^{(\gamma)}, \quad \mathcal{L}(\rho_\beta^{(gamma)})$$

Open problem:
Obtain mixing-time bounds for non-ergodic Davies generators.