bulk-to-boundary anyon fusion

from microscopic models

arXiv:2302.01835

joint work with
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Freie Universität Berlin
2+1d topological theories with defects

macroscopic theory

“2-manifolds to state(space)”

“3-bordisms to linear maps”

Atiyah 1988
2+1d topological theories with defects

macroscopic theory

"2-manifolds to state(space)"

"3-bordisms to linear maps"

Atiyah 1988

can be extended with defects

anyons

domain walls

twist defects
2+1d topological theories with defects

macroscopic theory

“2-manifolds to state(space)”

“3-bordisms to linear maps”

Can be extended with defects

anyons

domain walls

twist defects

Atiyah 1988
2+1d topological theories with boundaries

boundary: 1+1d interface to trivial theory

“no non-trivial anyons”
2+1d topological theories with boundaries

boundary: 1+1d interface to trivial theory

anyons close to the boundary can…

...condense
2+1d topological theories with boundaries

boundary: 1+1d interface to trivial theory

anyons close to the boundary can…

…condense

…confine
2+1d topological theories with boundaries

boundary: 1+1d interface to trivial theory

anyons close to the boundary can...

...condense

...confine

...become identified

“no non-trivial anyons”
folding trick: domain walls as boundaries

domain wall: 1+1d interface between two (non-trivial) theories

anyon model A \iff \text{anyon model } B
folding trick: domain walls as boundaries

domain wall: 1+1d interface between two (non-trivial) theories

\[
\begin{cases}
\text{anyon model } A \\ \cong \\ \text{anyon model } B
\end{cases}
\sim
\begin{cases}
\text{anyon model } A \otimes B \\ \cong \\ \text{trivial}
\end{cases}
\]
folding trick: domain walls as boundaries

domain wall: 1+1d interface between two (non-trivial) theories

\[
\begin{align*}
\{ \text{anyon model } A \} & \leftrightarrow \{ \text{anyon model } B \} \\
\cong & \{ \text{anyon model } A \boxtimes B \} \leftrightarrow \{ \text{trivial} \}
\end{align*}
\]
folding trick: domain walls as boundaries

domain wall: 1+1d interface between two (non-trivial) theories

\[
\begin{align*}
\text{anyon model } A & \iff \text{anyon model } B \\
\text{anyon model } A \boxtimes B & \iff \text{trivial}
\end{align*}
\]

defined by behaviour of anyons close to domain wall

Kitaev Kong 1104.5047
this work: bulk-to-boundary anyon fusion in non-chiral 2+1D theories

fusion vertex: point (in spacetime) where bulk anyon $i$ gets mapped to boundary anyon $j$

$\quad i \mapsto \bigoplus_{j} m_{i,j} j \quad \text{and} \quad j \mapsto \bigoplus_{i} m_{i,j} i$

defines “tunneling”

$m_{i,j} \in \mathbb{Z}_{\geq 0}$

related to:
Carqueville, Runkel, Schaumann 1710.10214
Bridgeman Barter Jones 1806.01279, 1810.09469 and Bridgeman Barter 1901.08069, 1907.06692
this work: bulk-to-boundary anyon fusion in non-chiral 2+1D theories

**fusion vertex:** point (in spacetime) where bulk anyon $i$ gets mapped to boundary anyon $j$

(and vice versa…)

\[ i \mapsto \bigoplus_j m_{i,j} j \quad \text{and} \quad j \mapsto \bigoplus_i m_{i,j} i \]

\[ m_{i,j} \in \mathbb{Z}_{\geq 0} \]

1. framework to calculate $m_{i,j} \in \mathbb{Z}_{\geq 0}$
2. closed formula for twisted finite gauge theory models

related to:

Carqueville, Runkel, Schaumann 1710.10214
Bridgeman Barter Jones 1806.01279, 1810.09469 and
Bridgeman Barter 1901.08069, 1907.06692
(further) contents

• **microscopic** fixed-point **models** with boundaries
• bulk and boundary **anyons**
• bulk-to-boundary **fusion vertex**
• calculating **fusion multiplicities**
• **applications**
• **conclusion** and outlook
fixed-point models

idea

define theory in terms of microscopic model with exact topological invariance
fixed-point models

idea

define theory in terms of microscopic model with exact topological invariance

defects implemented by modifying model along submanifold
fixed-point models

bulk

2D “string-net” models defined on branched triangulation
fixed-point models

bulk

2D “string-net” models defined on **branched triangulation**

states embedded into **finite-dim. tensor product space**

\[ \mathcal{H} = \bigotimes_{e} \text{span}_{\mathbb{C}} \{1, i, j, k, \ldots \} \]

with **local constraints** @ each triangle

\[ i \times j = \bigoplus_{k} N_{ij}^{k}, \quad N_{ij}^{k} \in \mathbb{Z}^{+} \]
fixed-point models

bulk

2D “string-net” models defined on branched triangulation

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\[ i \times j = \bigoplus_k N_{ij}^k k, \quad N_{ij}^k \in \mathbb{Z}^+ \]
fixed-point models

bulk

Example 1

finite group $G = \{1, g, h, k, \ldots \}$ defines

$$g \times h = gh \implies N_{g,h}^k = \delta_{k,gh}$$

e.g. $G = \mathbb{Z}_2 = \{0, 1\}$:

![Diagrams of triangulation and 2D "string-net" models defined on branched triangulation.](image)
fixed-point models

bulk

Example 1
finite group $G = \{1, g, h, k, \ldots \}$ defines

$$g \times h = gh \quad \Rightarrow \quad N^k_{g,h} = \delta_{k,gh}$$

e.g. $G = \mathbb{Z}_2 = \{0,1\}$:

Example 2
beyond group: $\{1, \tau\}$ with

$$\tau \times \tau = 1 + \tau$$
fixed-point models
bulk
topological invariance by identifying states on equivalent triangulations
related by Pachner moves
fixed-point models
bulk
topological invariance by identifying states on equivalent triangulations
related by Pachner moves
induce linear maps on associated spaces
e.g.
"F-symbol/tensor"
fixed-point models

bulk

topological invariance by identifying states on equivalent triangulations

related by Pachner moves

induce linear maps on associated spaces
e.g. "F-symbol/tensor"

building block of 3d triangulation
fixed-point models
bulk

sequence of Pachner moves is not unique
fixed-point models

bulk

sequence of Pachner moves is not unique!

F-symbols obey consistency condition

\[
F_{abc}^{ebe} F_{cda}^{hd} = \sum_k F_{kec}^{he} F_{kbi}^{fgd} F_{eab}^{ihk}
\]

+ similar constraints for other branching structures
fixed-point models

bulk

sequence of Pachner moves is not unique!

consistent data \( \{1,i,j,\ldots\}, \{N_{ij}^k\}, \{F_{ijk}^{lmm}\} \)

defines spherical fusion category

F-symbols obey consistency condition

\[
F_{efab} F_{ghai} = \sum_k F_{cdg}^{hek} F_{fkd}^{ebi} F_{efab}
\]

+ similar constraints for other branching structures
**fixed-point models**

**bulk**

Example 1

finite group $G = \{1, i, j, k, \ldots \}$

together with 3-cocycle $\omega : G^3 \to \mathbb{C}^\times$ fulfilling

$$\omega(ab, c, d)\omega(a, b, cd) = \omega(b, c, d)\omega(a, bc, d)\omega(a, b, c)$$

defines spherical fusion category $\text{Vec}^\omega(G)$

consistent data $\left\{\{1, i, j, \ldots \}, \{N_{ij}^k\}, \{F_{ijk}^{lmn}\}\right\}$

defines spherical fusion category

$$F_{ebc}^{fda}F_{hai}^{fgd} = \sum_k F_{cdg}^{hek}F_{gkc}^{kbi}F_{ihk}^{ebc}$$

+ similar constraints for other branching structures
anyons in the bulk
tube algebra
anyons in the bulk
tube algebra

bulk line defect described by $S^1 \times [0,1] =: T$
anyons in the bulk tube algebra

bulk line defect described by $S^1 \times [0,1] =: T$

microscopic model:

$V_T = \text{span}_\mathbb{C} \{ a, b, c, d \} \mapsto V_T \simeq \mathbb{C}^{D_T}$
anyons in the bulk

bulk line defect described by $S^1 \times [0,1] =: T$

microscopic model:

$$V_T \cong \mathbb{C}^{D_T}$$

$$V_T = \text{span}_\mathbb{C} \left\{ a, b, c, d \right\}$$

$V_T$ is not only a vector space…

…but also an algebra
anyons in the bulk
tube algebra

multiplication $*$ on $V_T$ defined by linear map associated to
anyons in the bulk
tube algebra

multiplication $\ast$ on $V_T$ defined by linear map associated to
anyons in the bulk tube algebra

**multiplication** * on $V_T$ defined by **linear map** associated to

\[(a, b, c, d)_T \ast (a', b', c', d')_T = \delta_{d,a'} \sum_{x, y} F_{yac}^{bb'x} F_{ybx}^{b'd'c'} F_{ybc}^{a'b'c'} (a, x, y, d')_T\]

as an algebra, $V_T = \bigoplus_i T_i$ with $T_i$ irreducible subspaces
anyons in the bulk
tube algebra

\[ V_T = \bigoplus T_i \text{ with } T_i \text{ irreducible subspaces} \]
anyons in the bulk tube algebra

\[ V_T = \bigoplus_i T_i \text{ with } T_i \text{ irreducible subspaces} \]

\[
\{ T_i \text{ irreducible} \} \xleftrightarrow{1-1} \{ c_i \in Z(V_T) : c_i^* c_j = \delta_{i,j} c_i \text{ indecomposable} \} \]

indecomposable central idempotents

c_i \text{ can be viewed as projector onto } T_i

decomposable:
\[ \exists \text{ central idempotents } a, b : c = a + b \]
anyons in the bulk
tube algebra

Example 1

$\text{Vec}^{\omega}(G)$ finite group $G$ and (normalized) 3-cocycle $\omega$

$$(g', h')_T * (g, h)_T = \delta_{g', hgh^{-1}} \beta(g', h)(g, h'h)_T$$

with $\beta(g, h) = \omega(h'hg(h'h)^{-1}, h', h) \omega(h', h, g) \omega(h', hgh^{-1}, h)$

irreps 1-1 with $(c, \rho_c)$ and associated central idempotents are

$$c_{(c, \rho_c)}^T = \frac{\dim(\rho_c)}{|Z(c)|} \sum_{g \in c} \sum_{h \in Z(g)} \tilde{\chi}_\rho^g(h)(g, h)_T$$

conjugacy class: $c$

$\beta_g$-projective irrep of centralizer $Z(c)$: $\rho_c$
fixed-point models

boundary

modify triangulation@ boundary: additional degrees of freedom on boundary vertices

states embedded into finite-dim. tensor product space

\[ \mathcal{H} = \bigotimes \text{span}\{1,i,j,k,\ldots\} \bigotimes \text{span}\{\alpha,\beta,\ldots\} \]
fixed-point models
boundary

modify triangulation@ boundary: additional degrees of freedom on boundary vertices

states embedded into finite-dim. tensor product space

\[ \mathcal{H} = \bigotimes \text{span}\{1, i, j, k, \ldots \} \bigotimes \text{span}\{\alpha, \beta, \ldots \} \]

with additional local constraints @ each boundary edge

\[ i \triangleright \alpha = \bigoplus_{\beta} M_{i,\alpha}^{\beta} \beta, \quad M_{i,\alpha}^{\beta} \in \mathbb{Z}^+ \]

consistent with bulk \[ N_{ij}^k \]

\[ i \triangleright (j \triangleright \alpha) \simeq (i \times j) \triangleright \alpha, \quad \forall i, j, \alpha \]
fixed-point models
boundary

modify triangulation@ boundary: additional **degrees of freedom** on boundary vertices

**Example 3 standard boundary**

the simples of any bulk model \{1,i,j,k,\ldots \} with

\[ i \triangleright j = i \times j \quad \forall i,j \]

define a consistent boundary model

\[
i \triangleright \alpha = \bigoplus_{\beta} M_{i,\alpha}^\beta \beta, \quad M_{i,\alpha}^\beta \in \mathbb{Z}^+\]

consistent with bulk \( N_{ij}^k \)

\[ i \triangleright (j \triangleright \alpha) \simeq (i \times j) \triangleright \alpha, \quad \forall i,j,\alpha \]
fixed-point models
boundary

topological invariance by adding moves at boundary
fixed-point models

boundary

topological invariance by adding moves at boundary

induce linear maps, represented by
fixed-point models

boundary

topological invariance by adding moves at boundary

induce linear maps, represented by

\[ L_{\delta d}^{\alpha e} L_{\delta c}^{\beta} = \sum_f L_{\alpha f}^{\gamma \beta} L_{\alpha c}^{\delta d} F_{\beta}^{ba} \]

+ similar constraints for other branching structures
fixed-point models

boundary

topological invariance by adding moves at boundary

induce linear maps, represented by

equivalent sequences of moves

\[
L_{\alpha e a}^\delta L_{\beta c b}^\gamma = \sum_f L_{\alpha f a}^\gamma L_{\beta d f}^\delta F_{b a f}^{b a f}
\]

constraints on \( \left\{ \{ \alpha, \beta, \ldots \}, \{ M^\beta_{i a} \}, \{ L^\beta_{k a i j} \} \right\} \) for

bulk \( C \) defines \( C \)-module category \( \mathcal{M} \)

+ similar constraints for other branching structures
anyons at the boundary

semi-tube algebra
anyons at the boundary
semi-tube algebra

boundary line defect described by $[0,1] \times [0,1] =: S$
anyons at the boundary
semi-tube algebra

boundary line defect described by [0,1] × [0,1] =: S

microscopic model:

\[ V_S \cong \mathbb{C}^{D_S} \]

\[ V_S = \text{span}_\mathbb{C} \left\{ \begin{array}{c}
\delta \\
\alpha \end{array} \right\} \]
anyons at the boundary
semi-tube algebra

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$V_S$ is not only a vector space...

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multiplication $*$ on $V_S$ defined by linear map associated to
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semi-tube algebra

multiplication * on $V_S$ defined by linear map associated to

$$(\alpha', \beta', \gamma', \delta', a')_S * (\alpha, \beta, \gamma, \delta, a)_S = \delta_{\alpha', \delta} \delta_{\beta', \gamma} \sum_b L_{\beta a a}^{\gamma' \beta' b} L_{a a a}^{\delta' \alpha' b} (\alpha, \beta, \delta', \gamma', b)_S$$

as an algebra, $V_S = \bigoplus_j S_j$ with $S_j$ irreducible subspaces

associated to central idempotents
anyons at the boundary
semi-tube algebra

Example 1’

for bulk \text{Vec}(G), a subgroup \(H\) together with a 2-cocycle \(\psi : H^2 \rightarrow U(1)\) fulfilling

\[
\psi(a, b)\psi(ab, c) = \psi(a, bc)\psi(b, c)
\]

\(\alpha, \beta \in G/H\)

define valid boundary. Associated central idempotents of \(S = \text{span}_\mathbb{C}\{(\alpha, \beta, g)_S\}\) are

\[
c^S_{(x, \kappa_x)} = \frac{\dim(\kappa_x)}{|K_x|} \sum_{\alpha, \beta \in G/H} \sum_{g \in \text{Stab}_G((\alpha, \beta))} \overline{\chi^S_{\kappa_x}(g)(\alpha, \beta, g)_S}
\]

double coset: \(x\)

\text{projective Irrep of stabilizer group } \text{Stab}(x): \kappa_c
bulk-to-boundary fusion vertex

\[ i : \text{Irrep } T_i \subseteq V_T \]
\[ j : \text{Irrep } S_j \subseteq V_S \]

\[ m_{i,j} : \text{multiplicity of } T_i \otimes S_j \subseteq V_C \]
bulk-to-boundary fusion vertex

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\textbf{fusion vertex} is described by
bulk-to-boundary fusion vertex

\[ i : \text{Irrep} \ T_i \subseteq V_T \]
\[ j : \text{Irrep} \ S_j \subseteq V_S \]

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fusion vertex is described by...
bulk-to-boundary fusion vertex

$i$: Irrep $T_i \subseteq V_T$

$j$: Irrep $S_j \subseteq V_S$

$m_{i,j}$: multiplicity of $T_i \otimes S_j \subseteq V_C$

fusion vertex is described by

We get space

$$V_C = \text{span}_C \left\{ \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array} \right\}$$

with $V_T \triangleright$ and $\triangleleft V_S$ action,

$$\begin{array}{c}
\quad \\
\quad \\
\quad \\
\quad
\end{array} : = \begin{array}{c}
\quad \\
\quad \\
\quad \\
\quad
\end{array}$$
bulk-to-boundary fusion multiplicities

actions $\triangleright$ and $\triangleleft$ commute

$$V_C = \bigoplus_{i,j} m_{i,j} T_i \otimes S_j$$

“bi-representation” of algebras $V_T$ and $V_S$
bulk-to-boundary fusion multiplicities

actions $\triangleright$ and $\triangleleft$ commute

$$V_C = \bigoplus_{i,j} m_{i,j} T_i \otimes S_j$$

using central idempotents, we can project onto $m_{i,j} T_i \otimes S_j$

$$m_{i,j} = \frac{1}{\dim(T_i) \dim(S_j)} \text{Tr} \left( c_i^T \triangleright \bullet \triangleleft c_j^S \right)$$
bulk-to-boundary fusion multiplicities

actions $\triangleright$ and $\triangleleft$ commute

$$V_C = \bigoplus_{i,j} m_{i,j} T_i \otimes S_j$$

using central idempotents, we can project onto $m_{i,j} T_i \otimes S_j$

$$m_{i,j} = \frac{1}{\dim(T_i)\dim(S_j)} \text{Tr} \left( c^T_i \triangleright \triangleleft c^S_j \right)$$

Example 1: $Vec^\omega(G)$ with valid subgroup $H$ and 2-cocycle $\psi$

$$m_{(c,\rho_c),(x,\kappa)} = \frac{1}{|G|} \sum_{g \in c} \sum_{\alpha \in G/H} \sum_{h \in Z(g) \cap \text{Stab}_G((g\triangleright \alpha, \alpha))} \psi^\alpha(h, g) \psi^\alpha(g, h) \tilde{\chi}^g_{\rho_c}(h) \tilde{\chi}^{(g\triangleright \alpha, \alpha)}_{\kappa_x}(h)$$
applications

TQFTs and topological quantum error correction

- explicit construction of one more invariant for TQFTs with microscopic models
- restriction on trivial boundary anyon gives Lagrangian algebra object
  - algebra morphism calculated together with bulk anyon fusion vertex

- describing and designing QEC protocols based on topological codes with defects
  - code dimension
  - logical algebra
  - folding trick: interfaces between codes
  - …
applications
TQFTs and topological quantum error correction

• explicit construction of one more invariant for TQFTs with microscopic models
• restriction on trivial boundary anyon gives Lagrangian algebra object
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• describing and designing QEC protocols based on topological codes with defects
  • code dimension
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  • …

more ideas?
conclusion and outlook

- microscopic models for non-chiral topological theories with boundaries
- description of anyons in terms of tube algebra
- construction of bi-representation describing bulk-boundary fusion vertex
- explicit formula for $\text{Vec}^\omega(G)$ models for bulk-boundary fusion multiplicities

- describe existing protocol(s) involving twisted gauge theory models (ongoing)
- include algebra morphism for condensable object (ongoing)
- resolving similar fusion vertex of line to line-on-surface defect in 3+1d
- understand more general defects in 3+1d with microscopic model
Thank you!

Questions?