Non-Abelian topological Berry phases

J.-S. Xu, K. Sun, Y.-J. Han, C. F. Li, G.-C. Guo, JKP,...

Nature Commun. **7**, 13194 (2016)
Science Advances **4**, eaat6533 (2018)
PRX Quantum 2, 030323 (2021)
M. Malik, arXiv:2304.05286 (2023)

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Tübingen, November 2023

Popularity of non-Abelian anyons

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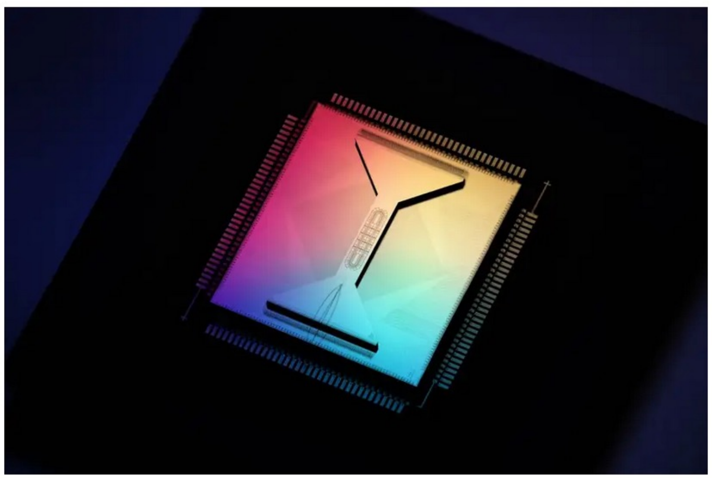
Physics

Weird particle that remembers its past discovered by quantum computer

Particles with unusual properties called anyons have long been sought after as a potential building block for advanced quantum computers, and now researchers have found one – using a quantum computer

By Alex Wilkins

9 May 2023



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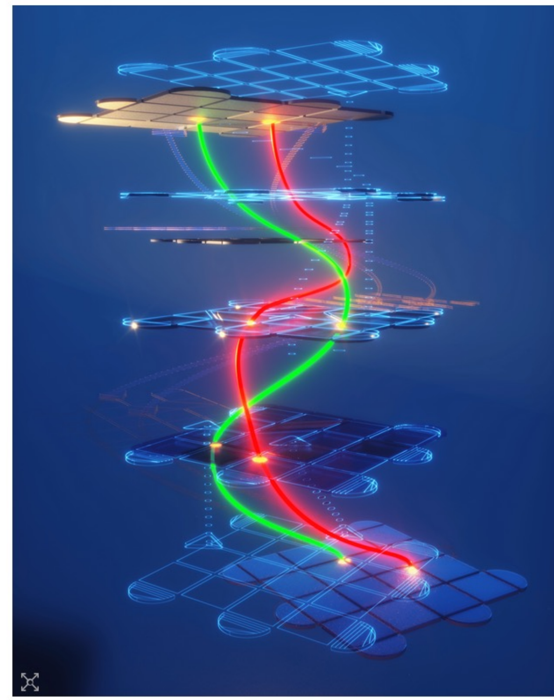
New type of quasiparticle emerges to tame quantum computing errors

22 May 2023

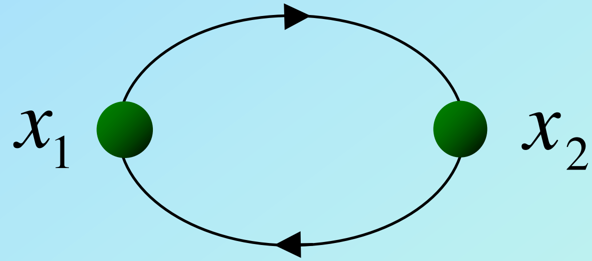


Topological manipulations: A graphic representing non-Abelian braiding of graph vertices in a superconducting quantum processor. (Courtesy: Google Quantum AI)

arXiv: 2305.03766, 2304.05286

Particle statistics

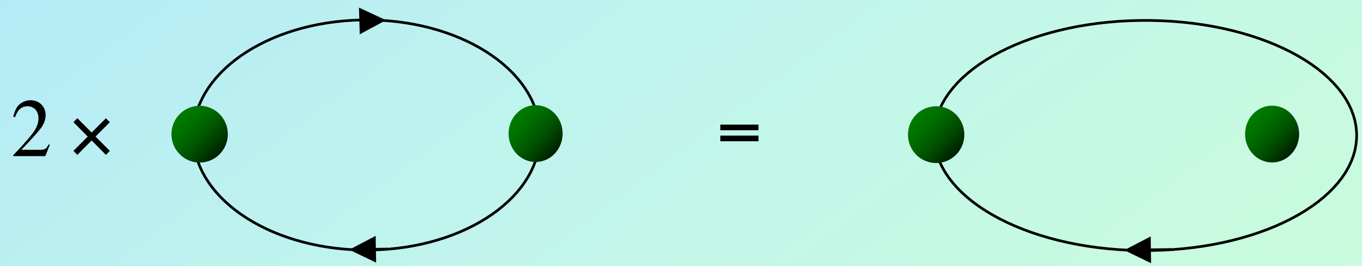
Exchange two identical particles:



Statistical symmetry:

Physics stays the same, but $|\Psi\rangle$ could change!

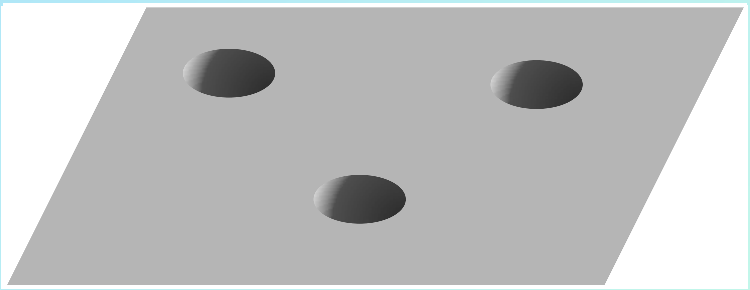
$$|\Psi(x_1, x_2)\rangle = ???|\Psi(x_2, x_1)\rangle$$



Overview

Superconducting Hamiltonians:

- Topological phase of matter
- **Majoranas – D(S_3)**



But SC are **tricky**:

- Non-conservation of particles
- Zero energy, localisation at boundary
- **Braiding** not possible (yet)

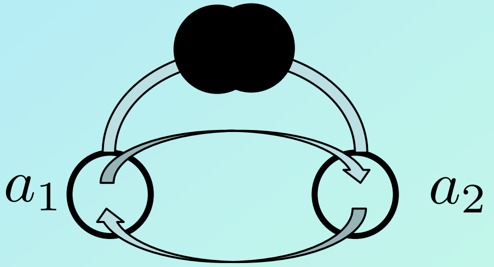
Spin-1/2 are easy to simulate (photons, atoms, ions, Josephson junctions, NMR,…)

We find **spin analogs of SC** and **simulate braiding.**

Superconducting fermion chain

Consider two sites with **tunnelling** and **pairing** interactions

$$H_{\rm SC} = -(\boxed{a_1^\dagger a_2 + a_2^\dagger a_1} + \boxed{a_1^\dagger a_2^\dagger + a_2 a_1})$$



Number of fermions is not conserved

due to pairing term.

Parity of fermions is conserved.

We will treat this Hamiltonian in two ways:

Fermions

Spins $\longleftrightarrow$ Majorana

From fermions to spins

Consider the Jordan-Wigner transformation:

$$a_i = \Big(\prod_{j<i} \sigma_j^z\Big)\sigma_i^+, \quad a_i^\dagger = \Big(\prod_{j<i} \sigma_j^z\Big)\sigma_i^-,$$

$$a_1 = \sigma_1^+, \quad a_2 = \sigma_1^z \sigma_2^+$$

Then we have:

$$H_{\mathrm{SC}} = \boxed{-(a_1^\dagger a_2 + a_1^\dagger a_2^\dagger + \mathrm{h.c.})} = -(\sigma_1^- \sigma_1^z \sigma_2^+ + \sigma_1^- \sigma_1^z \sigma_2^- + \mathrm{h.c.})$$

$$-\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_1 (\sigma_2^+ + \sigma_2^-) + \mathrm{h.c.} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_2 = \boxed{-\sigma_1^x \sigma_2^x}$$

The Ising Hamiltonian

The resulting spin Hamiltonian is

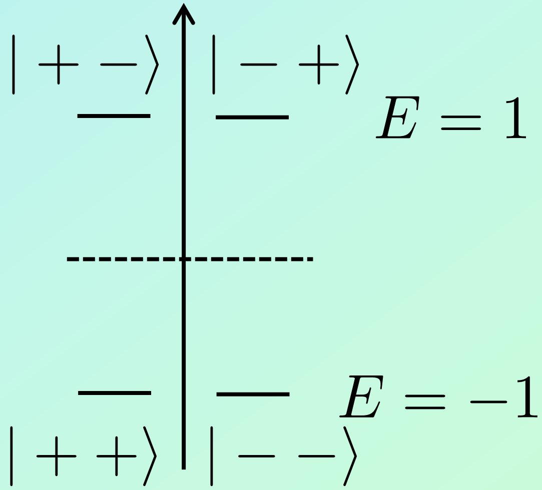
$$|+\rangle = \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)$$

$$H_{\text{spin}} = -\sigma_1^x \sigma_2^x$$

$$|-\rangle = \frac{1}{2}(|\uparrow\rangle - |\downarrow\rangle)$$

The ground and excited states are **doubly degenerate**

Use ground states to **encode a qubit:**

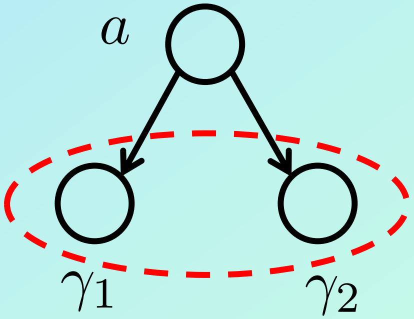


$$|\psi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L = \alpha|++\rangle + \beta|--\rangle$$

Majoranas from fermions

"Real" and "imaginary" decomposition gives Majoranas:

$$\gamma_1 = \frac{a + a^\dagger}{2}, \ \gamma_2 = \frac{a - a^\dagger}{2i}$$



They are fermions that **are their own anti-particles:**

$$\gamma_j^\dagger = \gamma_j$$

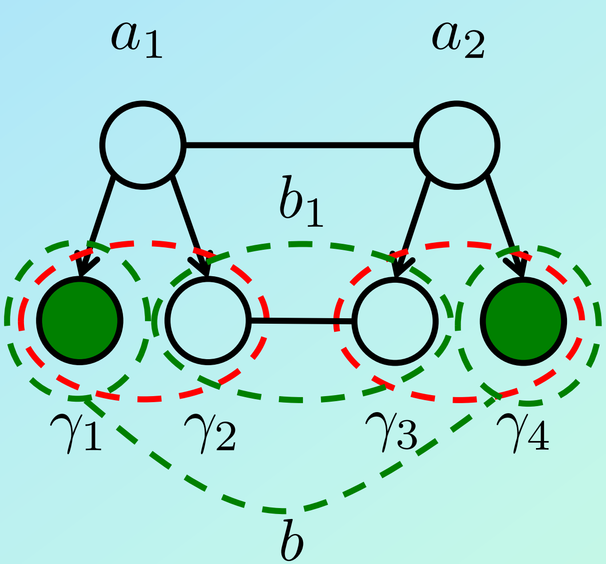
Up to now **Majoranas** are just a **mathematical construction.**

From fermions to Majoranas

Write superconducting Hamiltonian in **Majoranas**:

$$H_{\mathrm{SC}} = -\left(a_1^\dagger a_2 + a_1^\dagger a_2^\dagger + \mathrm{h.c.}\right) =$$

$$-(\gamma_1 - i\gamma_2)(\gamma_3 + i\gamma_4) - (\gamma_1 - i\gamma_2)(\gamma_3 - i\gamma_4) + \mathrm{h.c.} = -4i\gamma_2\gamma_3$$



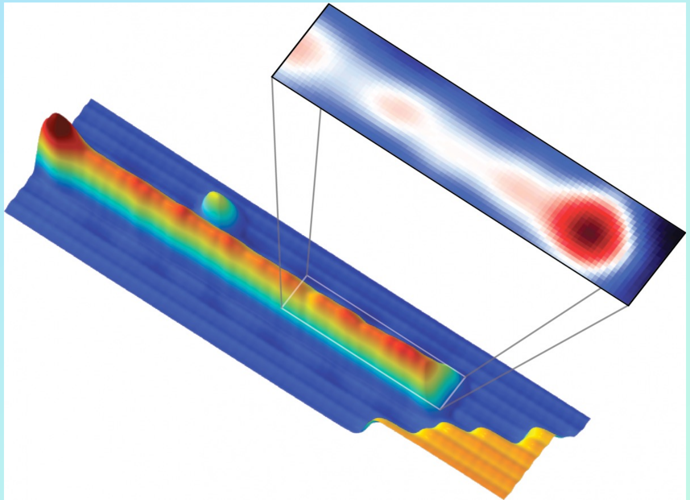
$$b_1 = \gamma_2 + i\gamma_3 \Rightarrow$$

$$H_{\mathrm{SC}} = -2(b_1^\dagger b_1 - \frac{1}{2})$$

$$b = \gamma_1 + i\gamma_4$$

QC: Manage expectations

- Tiny energy gap:
  - Temperature
- Finite extend:
  - Perturbations
  - Position inaccuracy
- Adiabatic transport
- State manipulations:
  - Preparation
  - Measurement



What are Majoranas?

Kitaev vs Ising

The JW trans is non-local.

$$a_i = \Big(\prod_{j<i}\sigma_j^z\Big)\sigma_i^+, \quad a_i^\dagger = \Big(\prod_{j<i}\sigma_j^z\Big)\sigma_i^-,$$

Both Hamiltonians are local.

Spectrum is the same: unitary time evolution operators are the same.

Eigenstates have different properties:

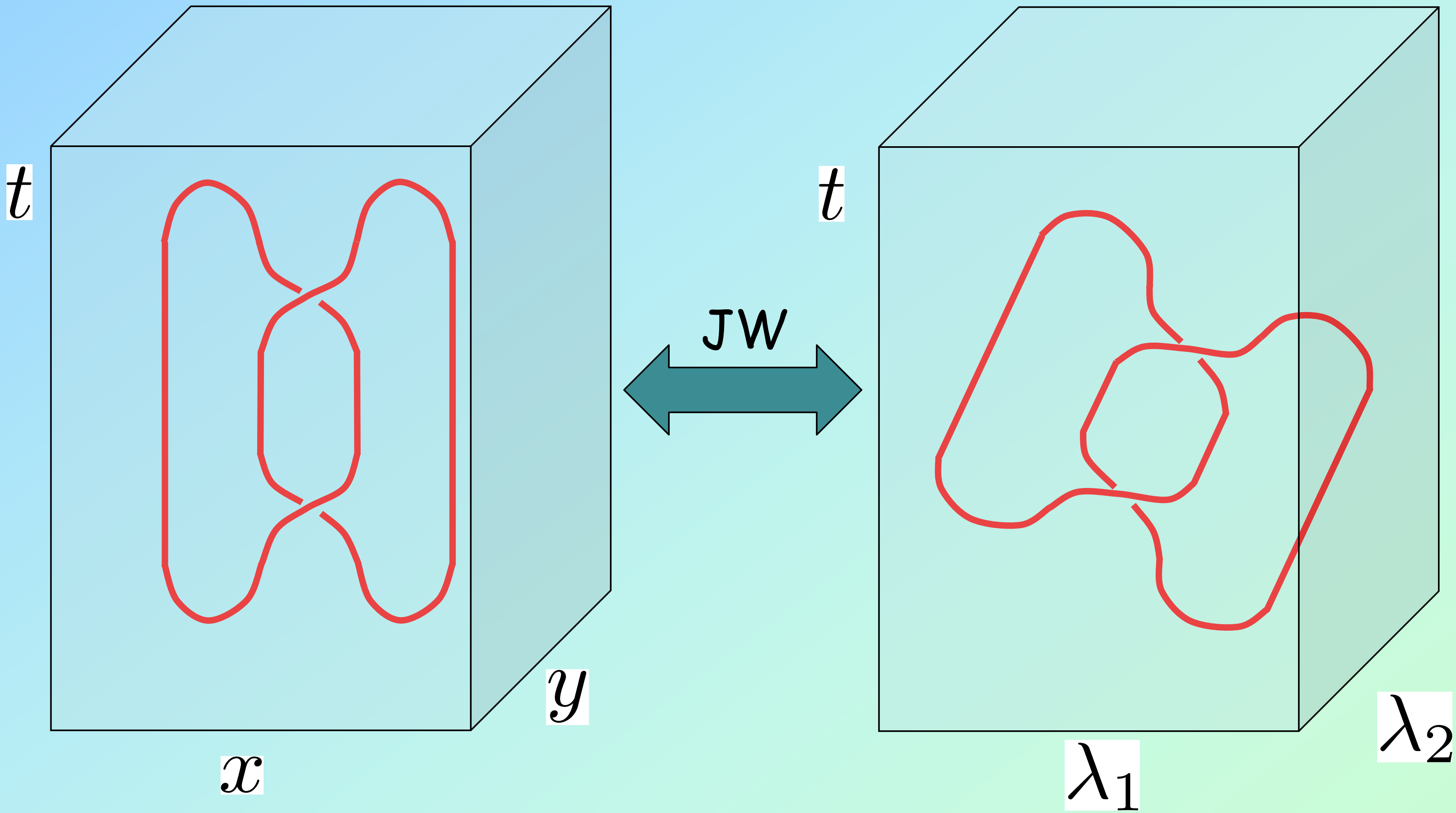
Local Majorana quasiparticles that do not overlap **map** to spin states with complete overlap.

Unitary mapping

$$H_{\rm spin} = U_{\rm JW} H_{\rm KCM} U^{\dagger}_{\rm JW}$$

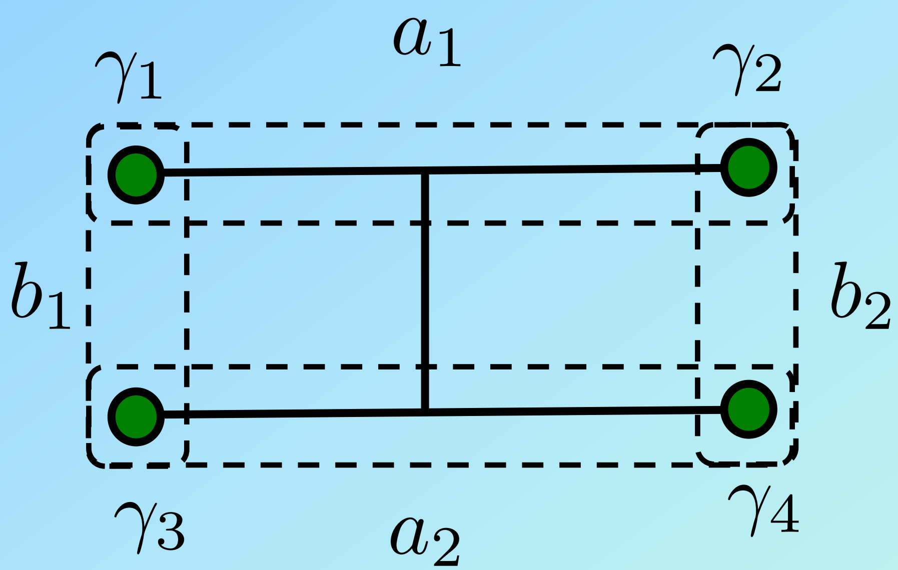
Majorana chain

Ising chain



Majoranas as anyons

Fusion and braiding of Majorana fermions



Fusion

$$\{a_1, b_1\} = \frac{1}{2}$$

One can show:

$$\Rightarrow |11\rangle_b = \sqrt{2} b_1^\dagger b_1 |00\rangle_a = \frac{1}{\sqrt{2}}(|00\rangle_a + |11\rangle_a)$$

Similarly:

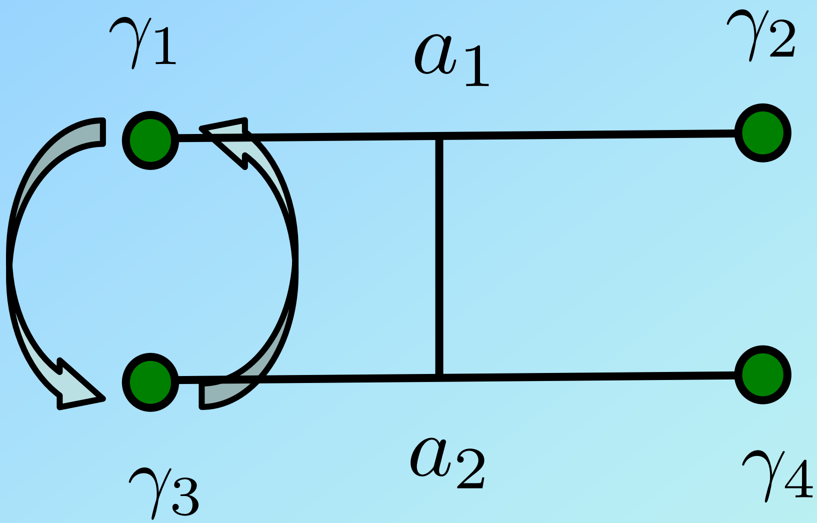
$$|00\rangle_b = -\frac{1}{\sqrt{2}}(|00\rangle_a - |11\rangle_a)$$

Bases related by

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Majoranas as anyons

Adiabatic braiding of Majorana fermions



Braiding

$$\mathcal{U} = a\mathbf{1} + b\gamma_1 + c\gamma_3 + d\gamma_1\gamma_3$$

$$\mathcal{U}^\dagger\mathcal{U} = 1,\ \mathcal{U}\gamma_1\mathcal{U}^\dagger \propto \gamma_3,\ \mathcal{U}\gamma_3\mathcal{U}^\dagger \propto \gamma_1$$

This gives two possible solutions

$\mathcal{U}^2 = e^{i\pi}$ ← *Nature Commun.*

$\mathcal{U}^2 = e^{i\pi/4}(a_1a_2 + a_1a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)$ ← *Science Advances*

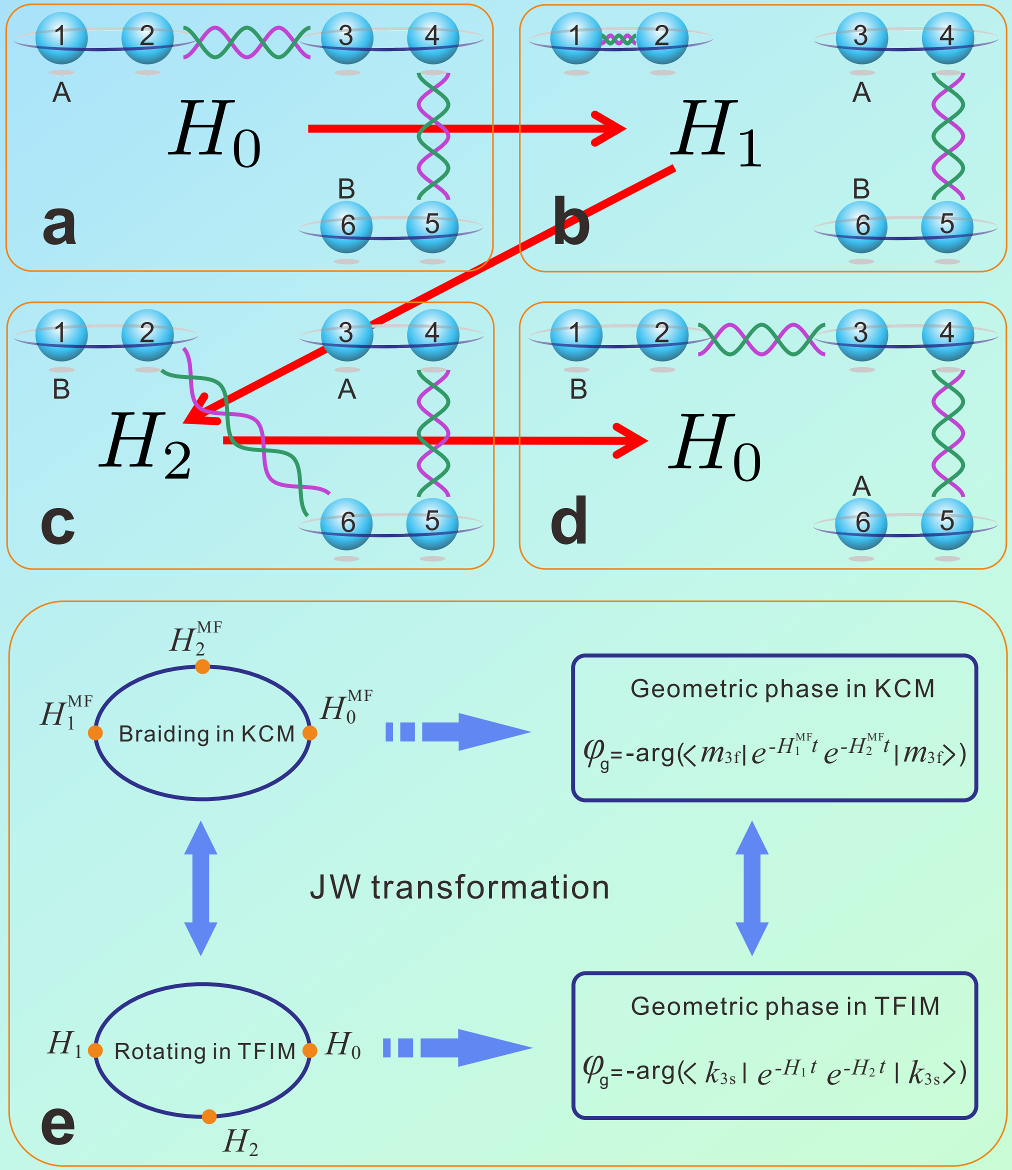
Photonic quantum simulator

Produce geometric phases:

Adiabatically change Hamiltonians -> Majoranas A and B are **exchanged.**

Translate Majoranas to spins: JW transf.

Do spin adiabatic evolution.



Photonic quantum simulator

Adiabatic dissipative evolution:

$$\varphi_{\mathrm{g}} = -\arg(\langle m_{Lf}|P_1 P_2 \cdots P_n|m_{Lf}\rangle)$$

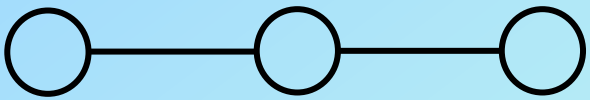
P_j project the state to the eigenstate of H_j

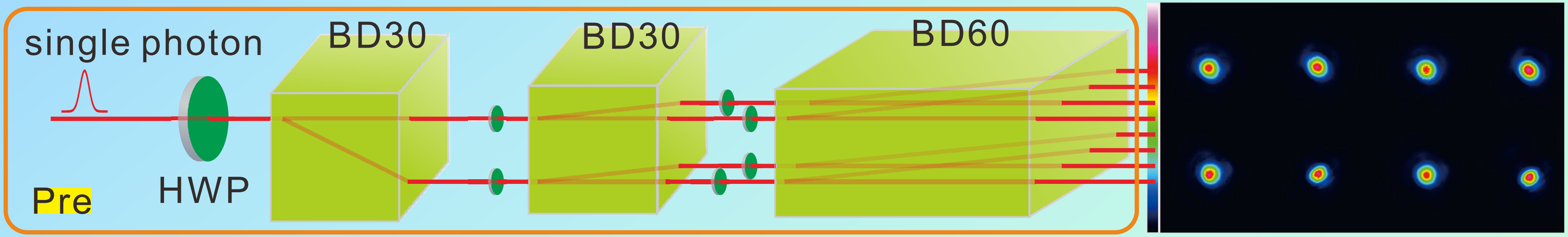
Can take $P_j \approx e^{-H_j t}$ for large t.

"Imaginary-time evolution"

Photonic quantum simulator

Three spins: $2^3 = 8$ states: $|\Psi\rangle = \sum_{j=1}^{8} c_j |j\rangle$





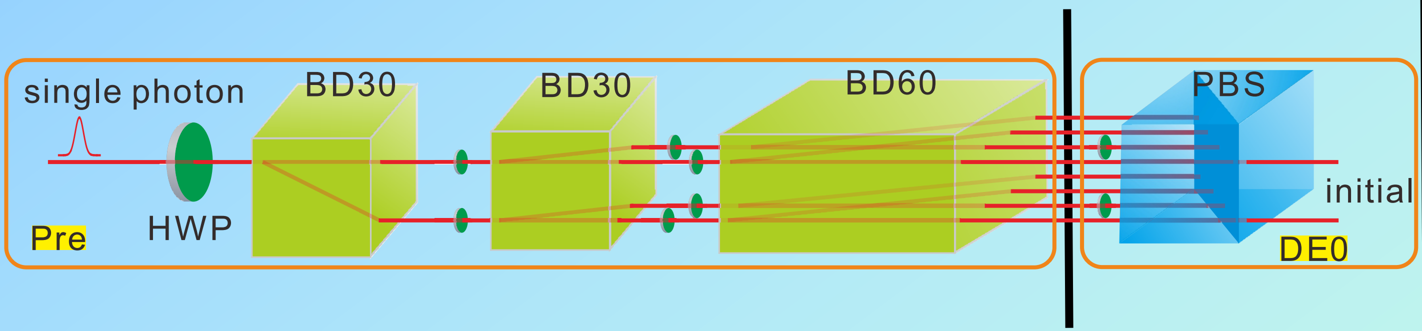
Pre: State preparation
HWP: Half Wave Plate
BD: Beam Displacer 30 or 60 mm

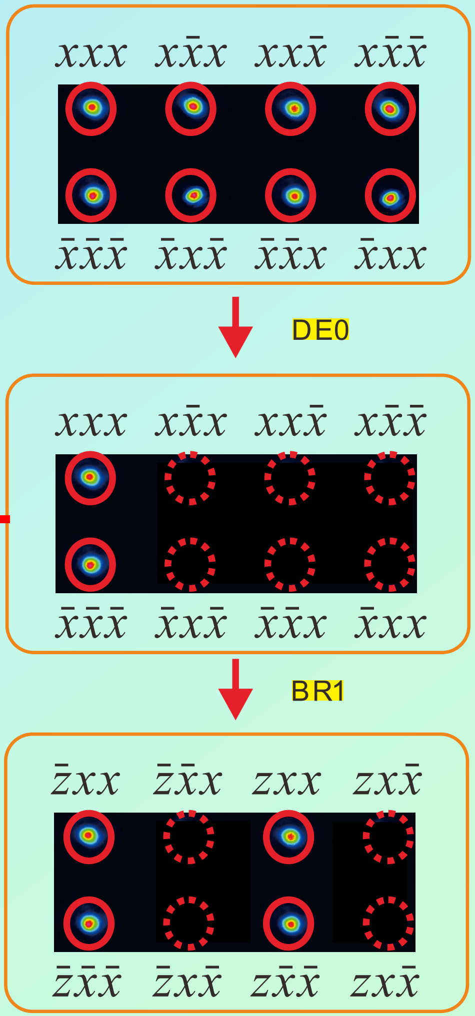
Use photonic mode for spin state
Use polarisation to couple to the environment

Photonic quantum simulator

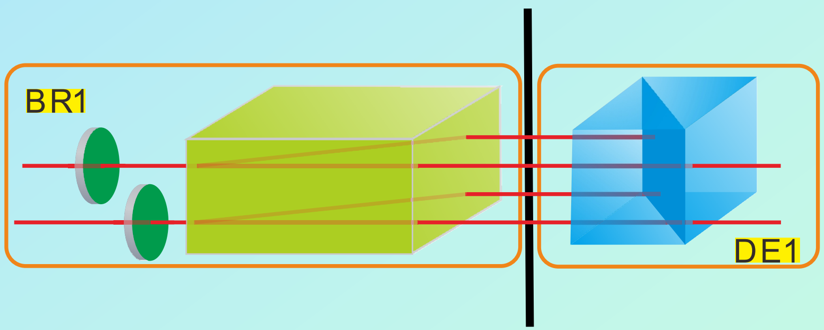
$$|\Psi\rangle = \sum_{j=1}^{8} c_j |j\rangle$$

Produce geometric phases:



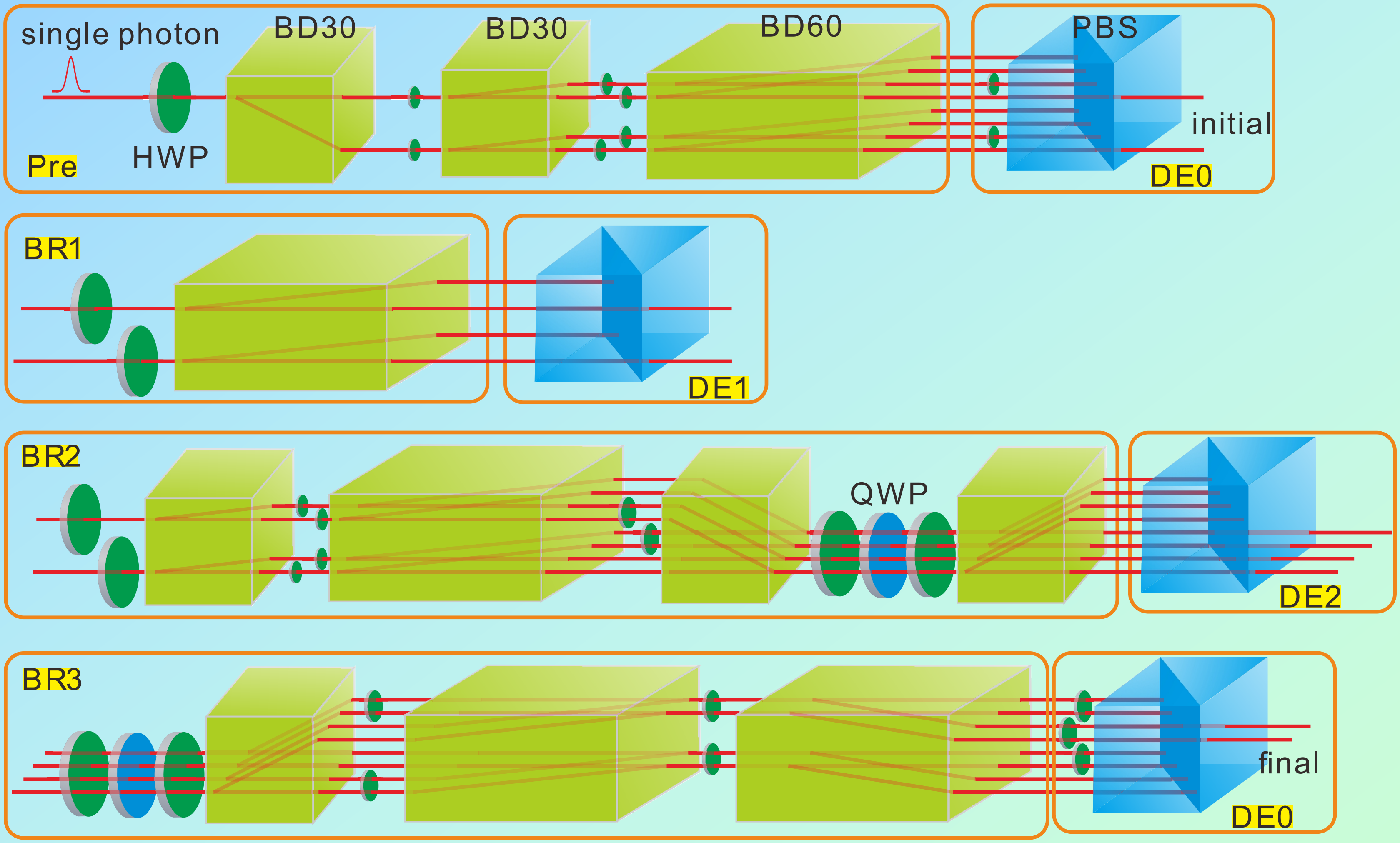


$$|\Psi\rangle = \alpha|xxx\rangle + \beta|\bar{x}\bar{x}\bar{x}\rangle$$



Photonic quantum simulator

Produce geometric phases:



Abelian Statistics

Experimentally produced geometric phases:

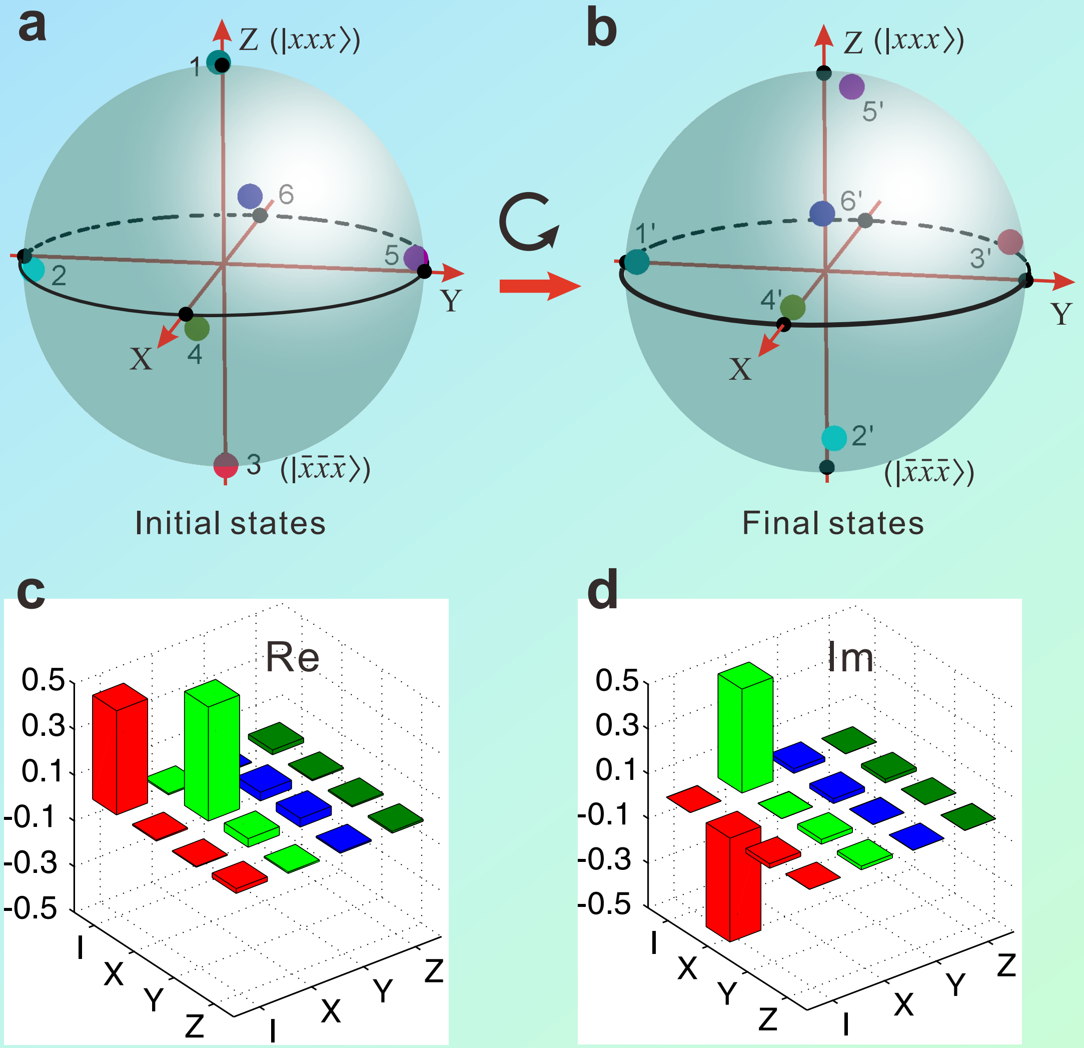
$$\varphi_g \approx \pi/2$$

Fidelity:

$$94.13 \pm 0.04\%$$

(errors deduced from Poissonian photon counting noise)

$$|\Psi\rangle = \alpha|xxx\rangle + \beta|\bar{x}\bar{x}\bar{x}\rangle$$

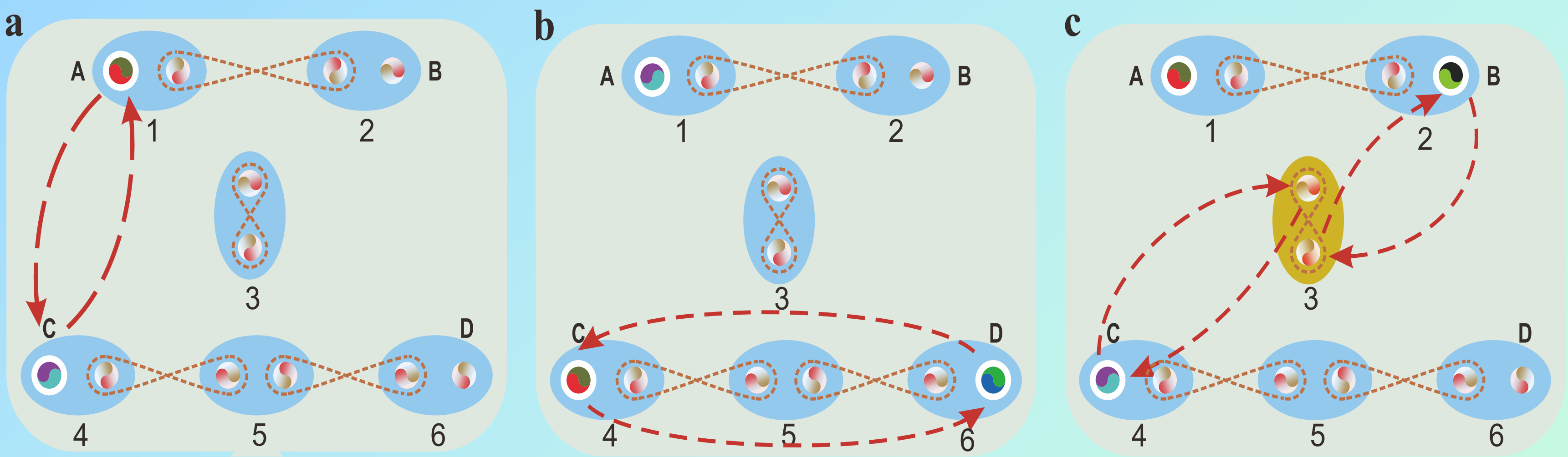


Tomography

[*Nature Commun.* 7, 13194 (2016)]

Non-Abelian Statistics

Exchange A and C Majorana fermions

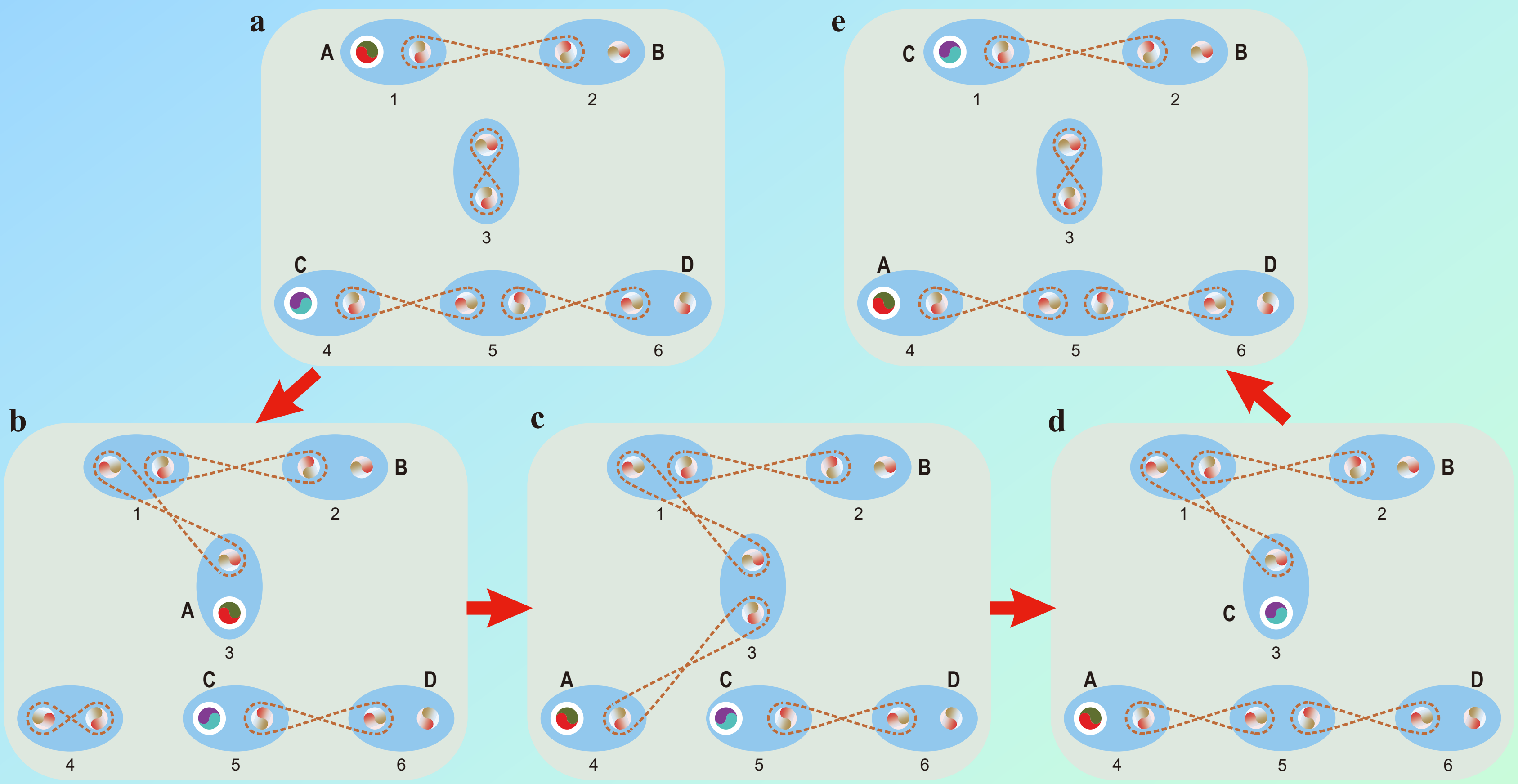


$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Non-Abelian statistics emerges as $HR \neq RH$

Non-Abelian Statistics

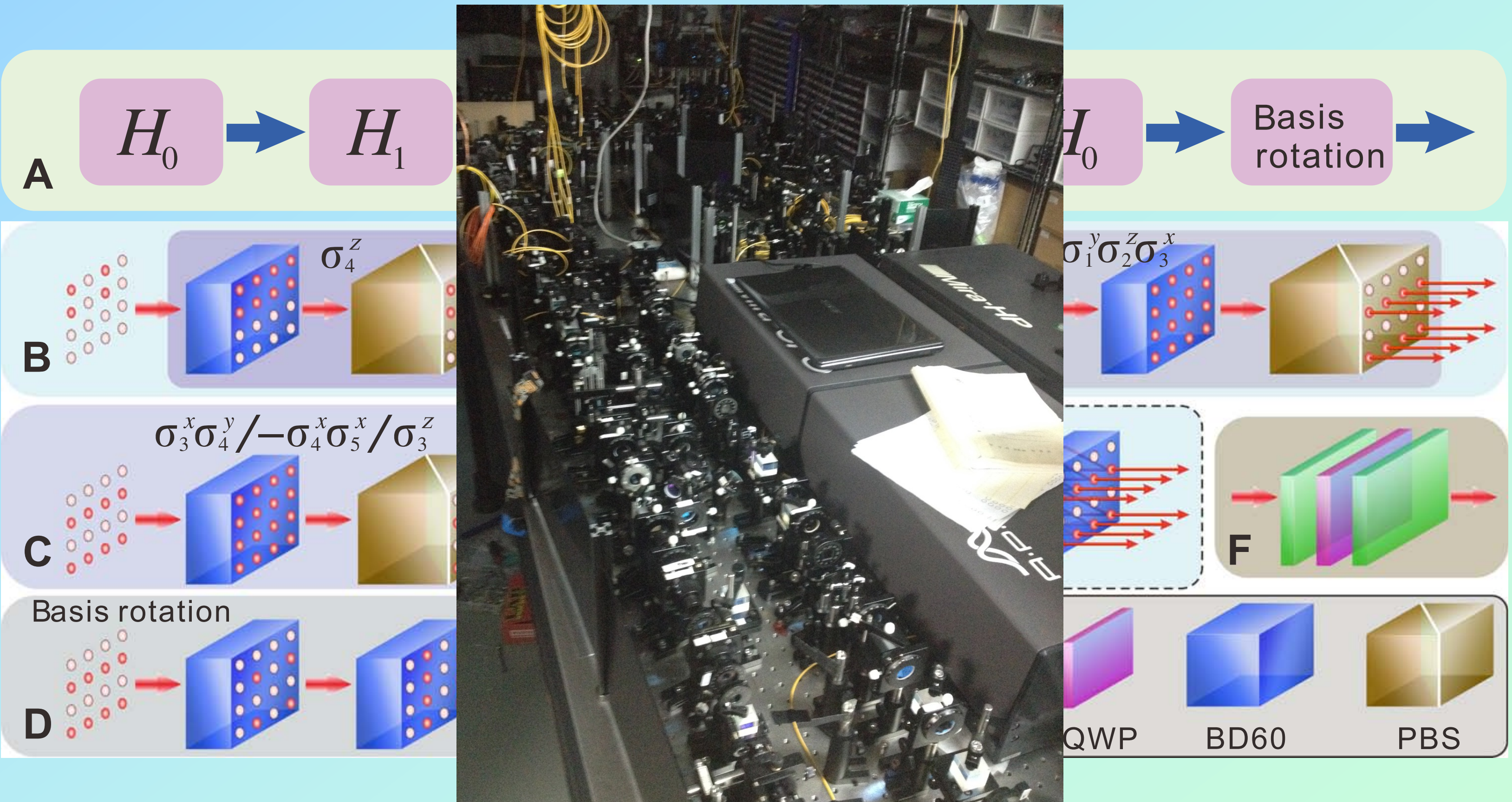
Exchange A and C Majorana fermions



Non-Abelian statistics emerges. $2^6 = 64$ states!

Non-Abelian Statistics

To implement it we use the following processes:



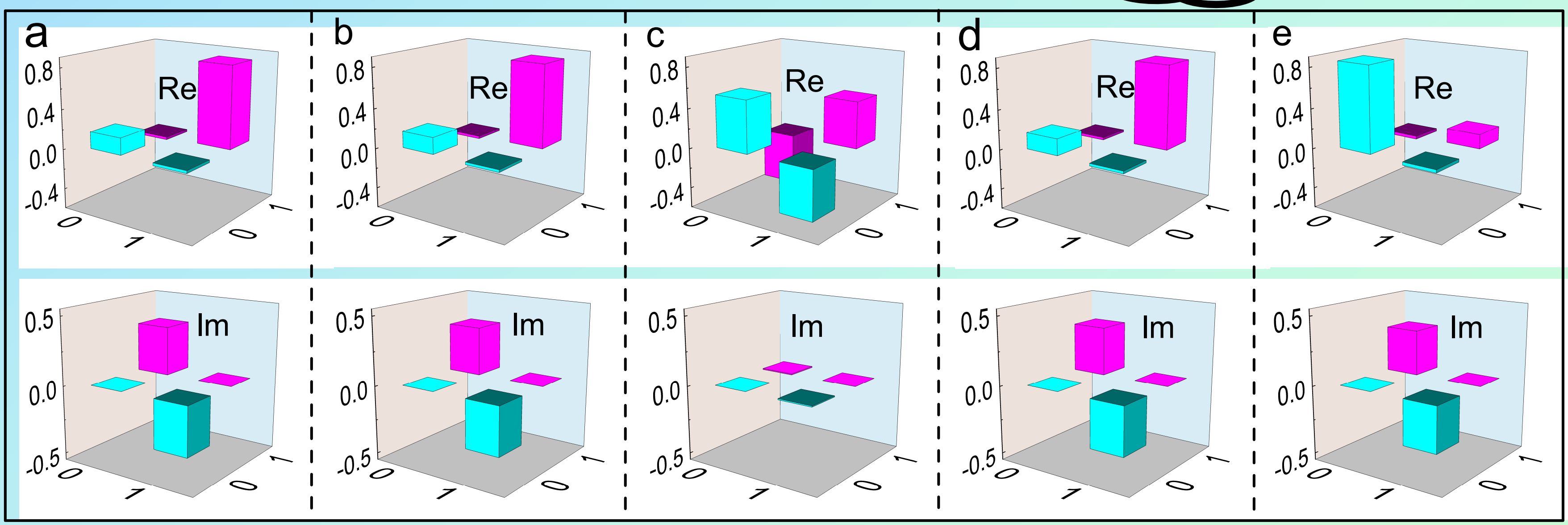
Non-Abelian Statistics

Fidelities:

Most gates F>97%
Total Fidelity >91%

Errors:

No errors Errors on 4 Errors on 3 4&5 E… …&4 Errors

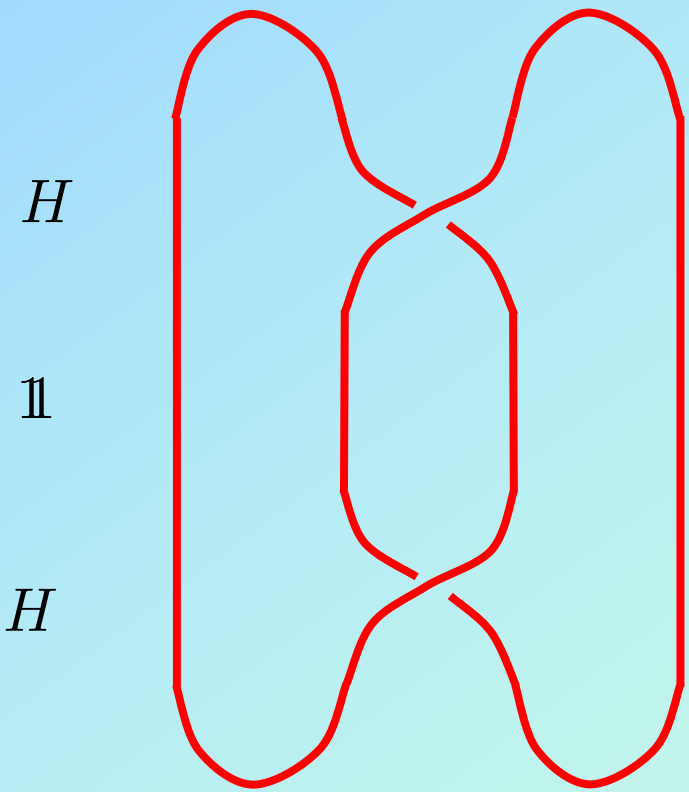


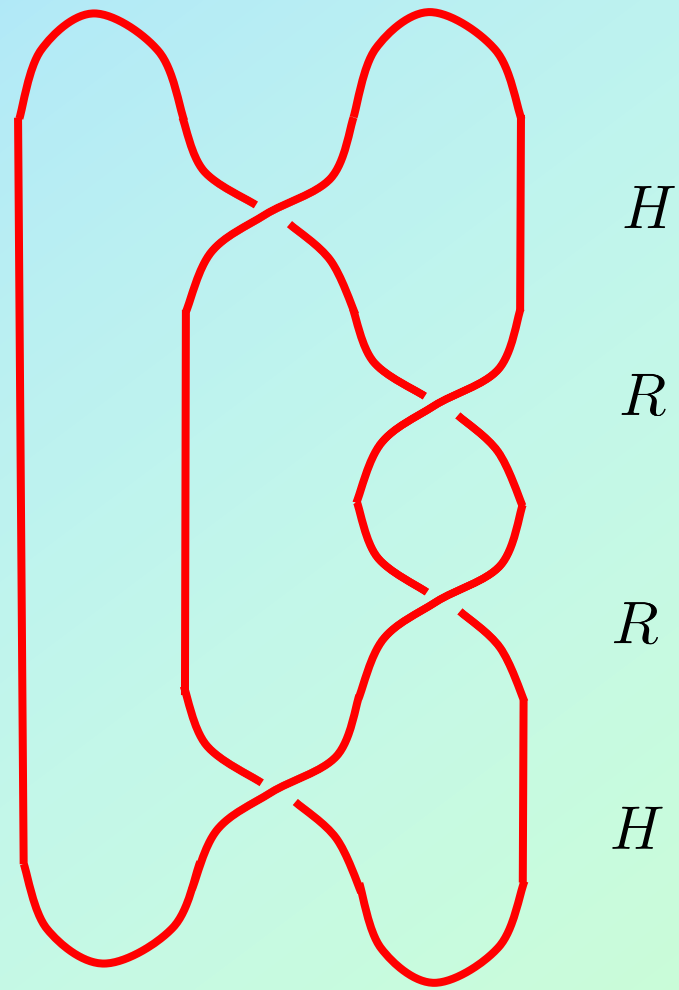
Algorithm

Deutsch-Jozsa Algorithm:

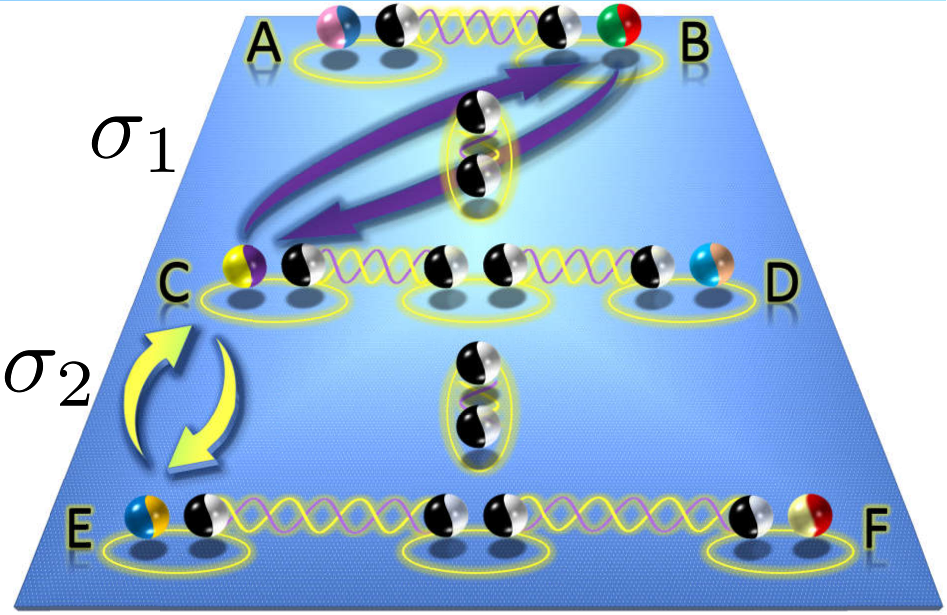
$$|0\rangle = HR^2H|0\rangle$$

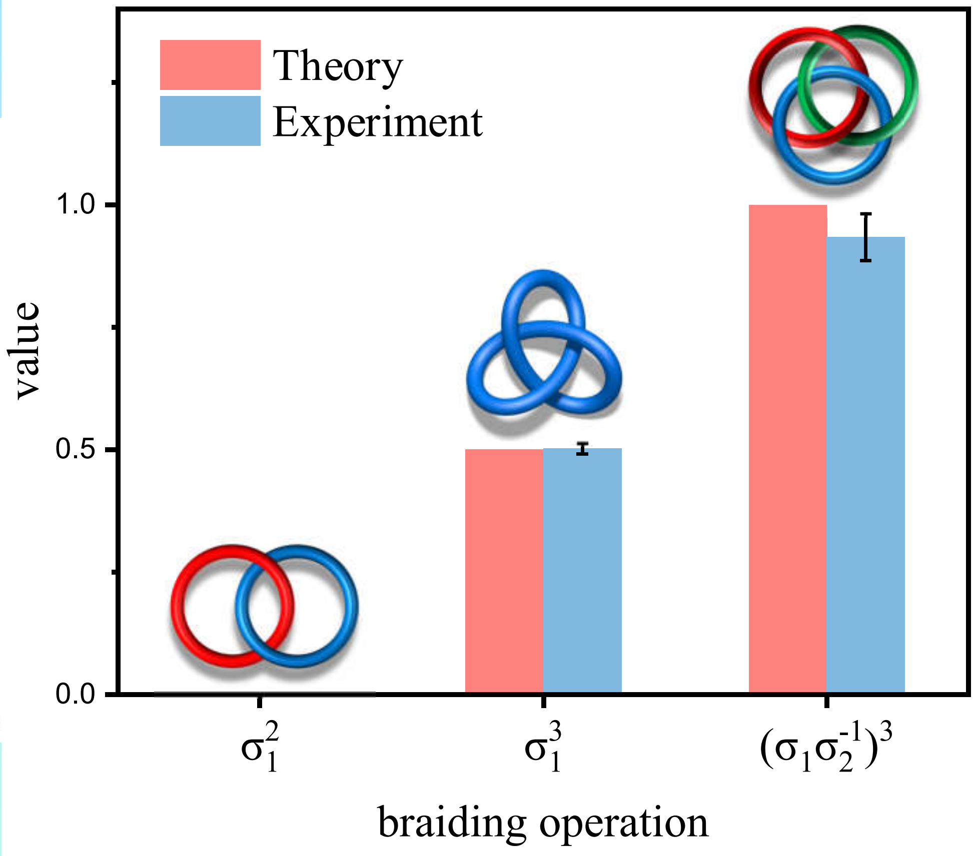
$$|1\rangle = H\mathbb{1}H|0\rangle$$





Scaling

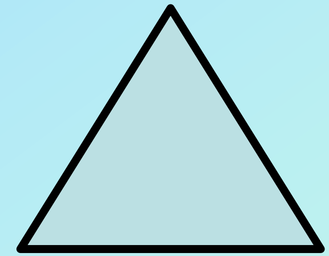




Using successive feedback of states.

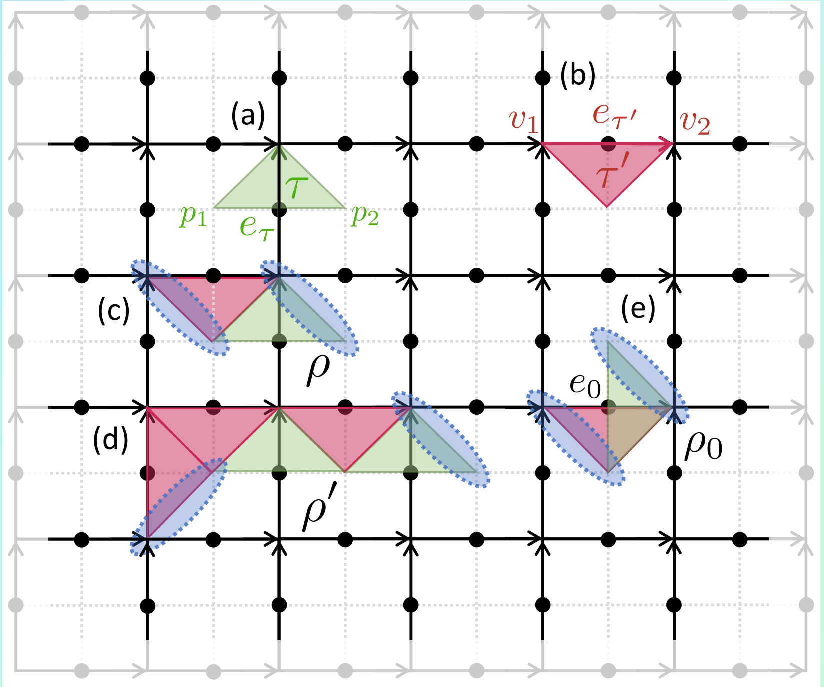
Non-Abelian D(S_3) Q. Double

$$S_3 = \{e, c, c^2, t, tc, tc^2\}$$



Anyons:

$$\{{\color{red}A}, {\color{red}B}, C, D, E, F, {\color{red}G}, H\}$$



Fusions:

$$B \times B = A,$$

$$B \times G = G,\ \ G \times G = A + B + G$$

Braiding:

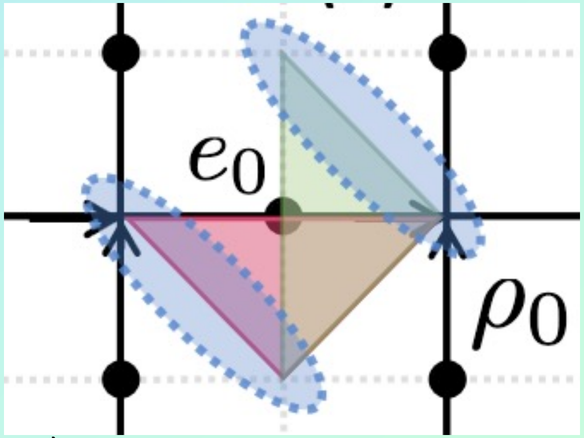
$$F^G_{GGG} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix},\ \ R^{GG} = \begin{pmatrix} \bar{\omega} & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

Non-Abelian D(S_3) Q. Double

Ribbon operators:

$$F_\tau^A = \boxed{|e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2|} + |t\rangle\langle t| + |tc\rangle\langle tc| + |tc^2\rangle\langle tc^2|$$
$$F_\tau^B = \boxed{|e\rangle\langle e| + |c\rangle\langle c| + |c^2\rangle\langle c^2|} - |t\rangle\langle t| - |tc\rangle\langle tc| - |tc^2\rangle\langle tc^2|$$

$$F_{\rho_0}^G = |c\rangle\langle e| + \omega|c^2\rangle\langle c| + \bar{\omega}|e\rangle\langle c^2| + \text{h.c.}$$



$$G \times G = \boxed{A + B} + G$$

$$F_{GGG}^G = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \quad R^{GG} = \begin{pmatrix} \bar{\omega} & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

Non-Abelian $D(S_3)$ Q. Double

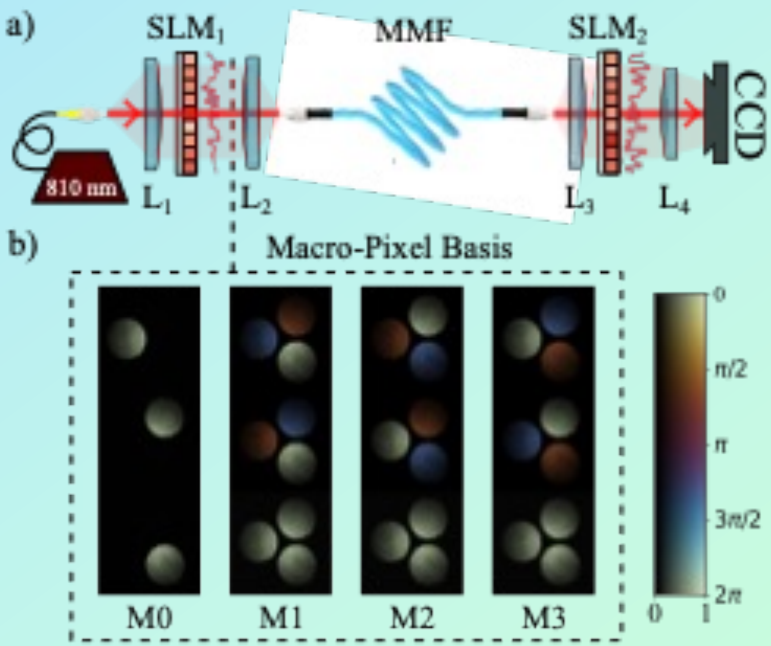
Braiding operations:

$$F^G_{\rho_0} F^G_{\rho_0} = F^A_{\rho_0} + F^B_{\rho_0} + F^G_{\rho_0}$$

$$F^G_{\rho_2} F^G_{\rho_1} = \bar{\omega}(F^A_{\rho_0} + F^B_{\rho_0}) + \omega F^G_{\rho_0}$$

$$F^G_{\rho_1} F^G_{\rho_2} = R^{GG} F^G_{\rho_2} F^G_{\rho_1}$$

Operation ($\mathbf{T}$)	Fidelity ($\mathcal{F}(\rho_{\mathcal{T}}, \rho_{\tilde{\mathcal{T}}})$)	Purity ($\mathcal{P}(\rho_{\tilde{\mathcal{T}}})$)
$F^G_{\rho_0}$	$95.23 \pm 0.93\%$	$96.04 \pm 0.03\%$
$F^G_{\rho_1} F^G_{\rho_2}$	$94.44 \pm 0.85\%$	$97.65 \pm 0.05\%$
$F^G_{\rho_2} F^G_{\rho_1}$	$97.59 \pm 0.59\%$	$94.43 \pm 0.06\%$



Summary

- Here we simulated their braiding properties, construct one-qubit gates, demonstrate fault-tolerance and a simple algorithm/Jones polynomials.

- Outlook:
  - Quantum algorithms
  - Quantum Machine Learning
  - Quantum error correction