MPS with measurement-assisted quantum circuits:
Transformations and phases of matter

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Motivation: Quantum state preparation

Given the description of a quantum state, how we create it on quantum hardware?
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Complexity of operations is restricted by noise and (often) locality.
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Given the description of a quantum state, how we create it on quantum hardware?

Quantum computing

Quantum simulation

Which states can be created with a reasonable amount of resources?

Complexity of operations is restricted by noise and (often) locality.
Topological phases, complexity, and state preparation

Topological phases is a classification according to entanglement complexity
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Topological phases is a classification according to \textit{entanglement complexity}

\[ |\Psi_1\rangle_N \sim |\Psi_2\rangle_N \]

Phase $= \text{Equivalence class}$

States in the same phase can be connected by a \textit{shallow-depth, local} quantum circuit
Topological phases, complexity, and state preparation

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Same phase implies “Roughly the same” circuit complexity

Hastings, Wen, PRB '05
Chen, Gu, Wen PRB ‘11
Haah et al. FOCS18
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States in the same phase can be connected by a **shallow-depth, local** quantum circuit

States in the **trivial** phase are **feasible** to prepare in a quantum simulator

Hastings, Wen, PRB ’05
Chen, Gu, Wen PRB ’11
Haah et al. FOCS18
Preparation of MPS

**MPS** are a fundamental part of many-body systems in 1D:

- Express *ground states* of local Hamiltonians
- Topological Phases *classification*
- Natural and efficient *language* for entanglement
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How can MPS be prepared?

**Sequential** preparation

- Applies to *every* MPS
- Depth $T = O(N)$
- Exact and explicit

Schön et al. PRA ‘07
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**Adiabatic** preparation

- *Injective* MPS (trivial phase)
- Depth $T = O(\text{Polylog}(N/\varepsilon))$
- Implicit (Hamiltonian simulation)

Ge et al. PRL ‘16
Bachmann et al. CMP ‘18
Preparation of MPS

What are the **limits** to MPS preparation?

- *What is the best possible scaling for injective MPS?*
- *How to achieve it?*
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Is it possible to **connect phases** without a blowup in the complexity?

- *How can measurements speed-up state preparation?*

Briegel et al., PRL ‘01  
Raussendorf et al, PRA ‘05  
Aguado et al., PRL ‘08  
Piroli et al., PRL ’21  
Tantivasadakarn et al., ’21  
Lu et al., PRXQ ‘22
Main Results

i) **Lower bound** on the complexity of preparing **injective** MPS:

- It is **impossible** to faithfully prepare any translational-invariant **injective** MPS over $N$ sites with a local quantum circuit of depth $o(\log N)$

  *(unless it is a product state)*
Main Results

i) **Lower bound** on the complexity of preparing **injective** MPS:

- It is **impossible** to faithfully prepare any translational-invariant **injective** MPS over N sites with a local quantum circuit of depth $o(\log N)$

  (unless it is a product state)

ii) Introduce an explicit algorithm for preparing **injective** MPS with the **optimal possible asymptotic scaling** $O(\log N)$

- Establishes the exact circuit complexity of injective MPS
- **Key technical tool is MPS renormalization**
Main Results

iii) Adapt the algorithm to include measurements. Then any MPS can be prepared in:

- $O(\log N)$ depth and 1-round of measurements, or
- $O(\log \log N)$ depth and $O(\log N)$ rounds of measurements
Lower bound for the preparation of injective MPS

\[ C(l) \sim \exp(-l/\xi) \]
Lower bound for the preparation of injective MPS

Target state
(injective MPS)

Finite depth circuit approximation
(strict light cone)

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\[ C(l > l_0) = 0 \]

Difference in correlations is key to prove complexity lower bound
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Difference in correlations is key to prove complexity lower bound

**Theorem.** Given

- a sequence of normal TI-MPS \( \{|\phi_N\rangle\}_N \) with nonzero correlation length, and
- a sequence \( \{|\psi_N\rangle\}_N \) of outputs from a local quantum circuit of depth \( T \) applies to product states, then:

If \( T = o(\log N) \), there exists \( N_0 \) such that for all \( N > N_0 \), we have

\[ 1 - |\langle \phi_N | \psi_N \rangle| > 1/2. \]
Preparing MPS with renormalization group (RG) transformation

**Goal:** Given an injective tensor A, prepare the corresponding MPS over N sites with \( O(\log N) \) depth
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(i) Blocking
Preparing MPS with renormalization group (RG) transformation

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(ii) Polar decomposition and approximate with the RG fixed point
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(iii) Approximate state

**Theorem:** Blocking $q = 2\xi \log(N)$ sites suffices to have a **vanishing error** in the thermodynamic limit
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How hard is to implement the isometry $V$?
Efficient implementation of the isometry

**Protocol:** Create entangled pairs and apply isometries

**Challenge:** Implement the isometry over output $\sim \log N$ sites with $O(\log N)$ depth
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**Sequential Scheme**

- Each $V$ has $O(\log N)$ depth
- Each gate has support over at most $dD^2$ sites
Efficient implementation of the isometry

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Challenge: Implement the isometry over output $\sim\log N$ sites with $O(\log N)$ depth

Sequential Scheme

- Each $V$ has $O(\log N)$ depth
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Tree Scheme

Each $V$ has:
- $O(\log \log N)$ layers (long-range gates)
- $O(\log N)$ depth for local gates
Exponential Speed-up using measurements

Injectivity implies short-range RG fixed-point: ...
Exponential Speed-up using measurements

Injectivity implies short-range RG fixed-point:

For a **general** tensor, RG fixed point has a **long-range** part:

\[
|\Omega\rangle_N = \sum_{j=1}^{b} a_j^{(N)} \otimes_{i=1}^{N/q} |\omega_j\rangle_{R_i L_{i+1}}
\]
Exponential Speed-up using measurements

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One round of measurements is enough to deterministically create the fixed point
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**Sequential Scheme**

Measurements are used only for the creation of the fixed point.

- Each isometry takes \(O(\log N)\) depth (no measurements)
- Depth \(O(\log N)\) with single round of measurements
Exponential Speed-up using measurements

Injectivity implies short-range RG fixed-point:

For a general tensor, RG fixed point has a long-range part:

\[ |\Omega\rangle_N = \sum_{j=1}^{b} a_j^{(N)} \sum_{i=1}^{N/q} \prod_{i} |\omega_j\rangle_{R_i L_{i+1}} \]

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**Sequential Scheme**
- Measurements are used only for the creation of the fixed point.
- Each isometry takes $O(\log N)$ depth (no measurements)
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**Tree Scheme**
- Use teleportation to implement long-range gates.
- Each layer now takes constant time.
- Depth $O(\log \log N)$ with $O(\log N)$ rounds of measurements

Lu et al., PRXQ '22
For translation-invariant MPS, correlation length is defined by the subleading eigenvalue of the transfer matrix $[\xi = -1/\ln(\lambda_2)]$. 

**Inhomogeneous MPS**
Inhomogeneous MPS

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For inhomogeneous MPS, injectivity alone does not guarantee finite correlation length
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For translation-invariant MPS, correlation length is defined by the subleading eigenvalue of the transfer matrix \( [\xi = -1/\ln(\lambda_2)] \).

For inhomogeneous MPS, injectivity alone does not guarantee finite correlation length.

“Definition” (Short-range correlated inhomogeneous MPS): A sequence of MPS \( \{ |\phi_N\rangle \}_N \) is short-range correlated if blocking \( q = O(\log N) \) sites the state can be well-approximated, up to local isometries, by:

\[
|\Omega\rangle_N = \bigotimes_{i=1}^{N/q} |\omega_j\rangle_{R_i L_{i+1}}
\]
Inhomogeneous MPS

Are “generic” inhomogeneous MPS short-range correlated?

Yes!

Garnerone, Oliveira, Zanardi, PRA ’10
Chen, Gu, Wen, PRB ’11
Haferkamp, Bertoni, Roth, Eisert, PRXQ ’21
Lancien, Pérez-García, AHP ’21
Haag, Baccari, GS, PRXQ ’23
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Haar random $D = 2$ MPS tensors

Error of the state drops exponentially in the size of block $q$ as in translation-invariant
Summary and outlook

- Measurements change the preparation complexity of MPS (and topological states in higher spatial dimensions!)

- Lower bound $\Omega(\log N)$ for the circuit complexity of injective MPS (i.e., trivial phase of 1D gapped local Hamiltonian ground states).

- Explicit algorithm to saturate the bound $\rightarrow$ optimal asymptotic scaling. Improves $O(\text{Polylog } N)$ depth (adiabatic preparation) to $O(\log N)$.

- Exponential speedup with measurements to depth $O(\log \log N)$ for any MPS.
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Future directions: PEPS?

- By blocking fixed point is approached “rapidly”
- Resulting isometry is “easy”
- Non-trivial class?
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Thank you!
Sequential generation of MPS

MPS in canonical form

Sequential circuit can create long-range correlations

Linear depth is necessary for GHZ and other long-range entangled MPS

However, ground states of 1D gapped local Hamiltonians (and injective MPS) have **exponentially-decaying correlations**

Bravyi et al., PRL '06