Pasting SPT phases in (1+1)d and (2+1)d lattice models

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based on ongoing work with

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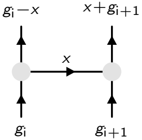
(1+1)d Rep(G)-symmetric lattice models

Throughout this talk: G finite abelian.

On PBCs: $\mathcal{H} = \bigotimes_{i=1}^{L} \mathbb{C}[G]$ for choice of $[\psi] \in H^2(G, \mathrm{U}(1))$:

$$\mathbb{h}_{i,n}^{\psi} := \sum_{g_i, g_{i+1}, x} h_n(g_i, g_{i+1}, x)$$



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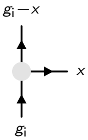
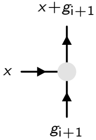


$= \psi(g_i - x, x)$,



$$= \frac{1}{\psi(x, g_{i+1})} .$$

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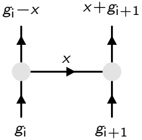
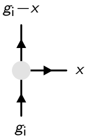
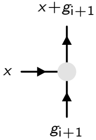
$$\mathbb{h}^{\psi}_{\mathsf{i},n} := \sum_{g_\mathsf{i}, g_{\mathsf{i}+1}, x} h_n(g_\mathsf{i}, g_{\mathsf{i}+1}, x)$$ 

, where 

$= \psi(g_\mathsf{i}-x, x)$, 

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Commute with symmetry operators labeled by $\chi \in \widehat{G} := \mathrm{Hom}(G, \mathrm{U}(1))$:

$$\mathbb{U}^{\chi} := \sum_{\{g_\mathsf{i}\}_\mathsf{i}} \Big(\prod_{\mathsf{i}} \chi(g_\mathsf{i}) \Big) |\{g_\mathsf{i}\}_\mathsf{i}\rangle\langle\{g_\mathsf{i}\}_\mathsf{i}| \, .$$

$\rightsquigarrow$ Generic Rep(G)-symmetric Hamiltonians $\mathbb{H}^{\psi} = \sum_\mathsf{i} \sum_n \mathbb{h}^{\psi}_{\mathsf{i},n}$.

Example: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)) \simeq \mathbb{Z}_2$ [Kennedy & Tasaki, 1992]

For **fixed** choice $h_1(g_i, g_{i+1}, (0,1)) = h_1(g_i, g_{i+1}, (1,0)) = -1$ $\forall$i, PBCs,

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$$\mathbb{H} = -\sum_{\mathrm{i}=1}^{L} \left[(X \otimes \mathbb{1})_{\mathrm{i}} (X \otimes \mathbb{1})_{\mathrm{i}+1} + (\mathbb{1} \otimes X)_{\mathrm{i}} (\mathbb{1} \otimes X)_{\mathrm{i}+1} \right], \quad \text{SSB, } \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ ferromagnetic order,}$$

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Result: distinct models that only differ in choice of inequivalent 2-cocycle (‘pasting SPT’) are **dual**.

[Lootens et al, 2021]

Dualities in (1+1)d [Lootens et al, 2021]

1. Local symmetric operators $\rightsquigarrow$ local symmetric operators ($\mathbb{h}^{\psi}_{i,n} \rightsquigarrow \mathbb{h}^{\psi'}_{i,n}$, $\mathbb{H}^{\psi} \rightsquigarrow \mathbb{H}^{\psi'}$).
2. Local non-symmetric operators $\rightsquigarrow$ non-local non-symmetric operators (ferromagnetic order operator $\rightsquigarrow$ string order operator).

Duality operator between $\mathbb{H}$ and $\mathbb{H}^{\psi}$ on PBCs labeled by $\pi \in \mathsf{Rep}^{\psi}(G)$: [Delcamp & BVDC, to appear]

$$\mathbb{O}^{\pi} := \sum_{\{g_i\}_i} \mathrm{tr}\Big(\prod_i \pi(g_i)\Big) |\{g_i\}_i\rangle\langle\{g_i\}_i| \quad \Longrightarrow \quad \mathbb{O}^{\pi}\, \mathbb{h}_{i,n} = \mathbb{h}^{\psi}_{i,n}\, \mathbb{O}^{\pi}, \quad \forall i, n.$$

Kinematical constraints due to PBCs, e.g. for $\mathbb{Z}_2 \times \mathbb{Z}_2$ both $\mathbb{H}$ and $\mathbb{H}^{\psi}$ in even $\mathbb{Z}_2 \times \mathbb{Z}_2$ sector!

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3. Isometry relating models (same spectra).

Other sectors accessible by considering **symmetry-twisted boundary conditions**.

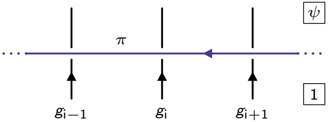
$\rightsquigarrow$ $\big(\bigotimes_{\mathsf{i}=1}^{L} \mathbb{C}[G]\big) \otimes \mathbb{C}[\widehat{G}]$ & modified local operators and symmetry operators.

$\rightsquigarrow$ (Simple) **Topological sectors** $(c, \eta) \in G \times \widehat{G}$.

Topological sectors [Lootens et al, 2022] [Delcamp & BVDC, to appear]

Duality operator on PBCs looks like:

$$\mathbb{O}^\pi := \sum_{\{g_i\}_i} \mathrm{tr}\Big(\prod_i \pi(g_i)\Big) \, |\{g_i\}_i\rangle\langle\{g_i\}_i| \equiv \sum_{\{g_i\}_i}$$



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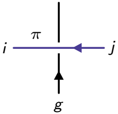

$$\equiv \pi(g)_{ij} \, |g\rangle\langle g| \otimes |\pi, i\rangle\langle \pi, j| \, .$$

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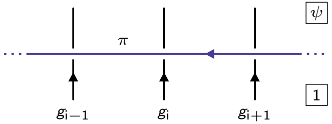

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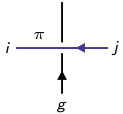
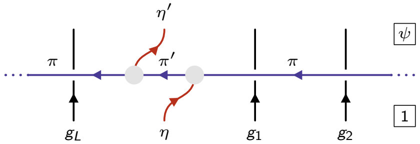


$$\equiv \pi(g)_{ij}\, |g\rangle\langle g| \otimes |\pi, i\rangle\langle \pi, j|\,.$$

On twisted BCs gets modified as:

$$\mathbb{O}^{\pi}_{\eta,\eta'} := \sum_{\{g_i\}_i}$$



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Leading to a permutation of topological sectors [Nikshych & Riepel, 2014] [Fuchs et al, 2015] [Li et al, 2023]

$$\mathrm{Aut}(G \times \widehat{G}) : (c, \eta) \mapsto (c, \eta^{\psi}_c)\,, \quad \text{where} \quad \frac{\psi(-, c)}{\psi(c, -)}\, \eta(-) \equiv \eta^{\psi}_c(-).$$

Generalization

Generalization to (2+1)d is systematic following the principle of **categorification**. [Delcamp & Tiwari, 2023] [Delcamp & BVDC, to appear]

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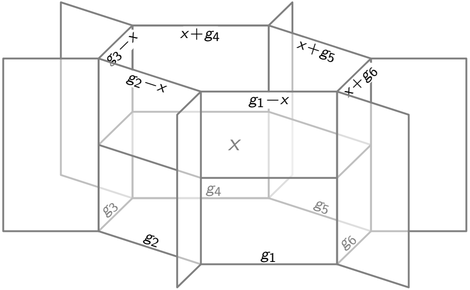
$$(1{+}1)\text{d SPTs} \leftrightarrow [\psi] \in H^2(G, \mathrm{U}(1)) \qquad (2{+}1)\text{d SPTs} \leftrightarrow [\alpha] \in H^3(G, \mathrm{U}(1))$$

Realization Hamiltonian in duality class	Module 1-category Vec^{ψ}	Module 2-category $2\mathsf{Vec}^{\alpha}$
Symmetry operators	$\mathsf{Fun}_{\mathsf{Vec}_G}(\mathsf{Vec}^{\psi}, \mathsf{Vec}^{\psi}) \simeq \mathsf{Rep}(G)$	$2\mathsf{Fun}_{2\mathsf{Vec}_G}(2\mathsf{Vec}^{\alpha}, 2\mathsf{Vec}^{\alpha}) \simeq 2\mathsf{Rep}(G)$
Duality operators	$\mathsf{Fun}_{\mathsf{Vec}_G}(\mathsf{Vec}, \mathsf{Vec}^{\psi}) \simeq \mathsf{Rep}^{\psi}(G)$	$2\mathsf{Fun}_{2\mathsf{Vec}_G}(2\mathsf{Vec}, 2\mathsf{Vec}^{\alpha}) \simeq 2\mathsf{Rep}^{\alpha}(G)$

(2+1)d 2Rep(G)-symmetric lattice models [Delcamp & Tiwari, 2023]

On hexagonal lattice, for choice of $[\alpha] \in H^3(G, \mathrm{U}(1))$:

$$\mathbb{h}^{\alpha}_{\mathsf{p},n} = \sum_{x} \sum_{\{g_i\}_i} h_{\mathsf{p},n}(\{g_i\}_i, x)$$



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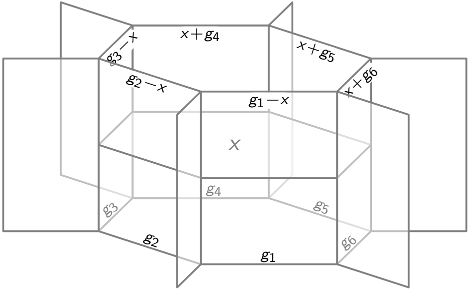

$\rightsquigarrow$ Generic 2Rep(G)-symmetric Hamiltonians $\mathbb{H}^{\alpha} = \sum_{\mathsf{p}} \sum_{n} \mathbb{h}^{\alpha}_{\mathsf{p},n}$:

- Simple objects = surface operators,
- 1-morphisms = topological line operators.

Implemented on the lattice by PEPO and MPO operators respectively.

Duality mapping

Result: distinct models that only differ in choice of inequivalent 3-cocycle ('pasting SPT') are **dual**.

Example: $G = \mathbb{Z}_2$, $H^3(G, \mathrm{U}(1)) \simeq \mathbb{Z}_2 \rightsquigarrow$ duality between toric code and double semion. [Levin & Wen, 2005]

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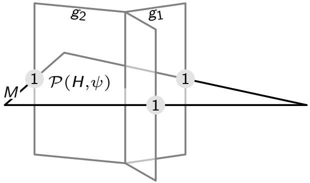
Duality operators labeled by simple objects $\mathcal{P}$ in $2\mathrm{Rep}^{\alpha}(G) \equiv \mathrm{Mod}(\mathrm{Vec}_G^{\alpha})$, in one-to-one correspondence with

- $H \leqslant G$,
- $\psi \in C^2(H, \mathrm{U}(1))$ s.t. $\mathrm{d}\psi = \alpha^{-1}|_{H \times H \times H}$.



Hilbert space decomposes in topological sectors $(g, h, \eta) \in G \times G \times \widehat{G}$.

Permutation: $\mathrm{Aut}(G \times G \times \widehat{G}) : (g, h, \eta) \mapsto (g, h, \eta^{\alpha}_{g,h})$, where $\frac{\alpha(g,h,-)\alpha(-,g,h)\alpha(h,-,g)}{\alpha(g,-,h)\alpha(h,g,-)\alpha(-,h,g)} \equiv \eta^{\alpha}_{g,h}(-)$.

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Reach out

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