Generalized string nets at finite temperature

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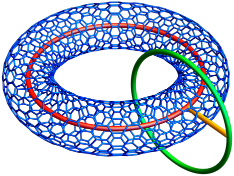

in collaboration with:

- Jean-Noël Fuchs and Anna Ritz-Zwilling (LPTMC, Paris)
- Steven H. Simon (Rudolf Peierls Centre for Theoretical Physics, Oxford)

J. Vidal, Phys. Rev. B **105**, L041110 (2022) / arXiv:2108.13425

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

Topological quantum order in condensed matter in three dates

- 1989 : High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, A. Zee)
- 1997 : Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005 : String-net condensation (M. Levin, X.-G. Wen)

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Concepts coming from 70's and 80's

- Lattice gauge theories (Wegner, Wilson, Kogut)
- Conformal field theories (Pasquier, Verlinde, Moore, Seiberg)
- Topological quantum field theories (Witten)
- Knot theory (Kauffman, Jones)

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In this talk:

Two-dimensional gapped quantum systems with:

- Topology-dependent ground-state degeneracy
- Anyonic excitations

Outline

Outline

The Levin-Wen model in a nutshell

The Levin-Wen (string-net) model

- Microscopic lattice model M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
- Trivalent graph (honeycomb lattice, two-leg ladder,...)

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Input: Unitary Fusion Category $\mathcal{C}$

- Set of objects (strings, labels, charges, superselection sectors, particles...)
- Fusion rules: $a \times b = \sum_c N_{ab}^c c \rightarrow$ Quantum dimensions
- Associativity of the fusion rules $\rightarrow$ F-symbols

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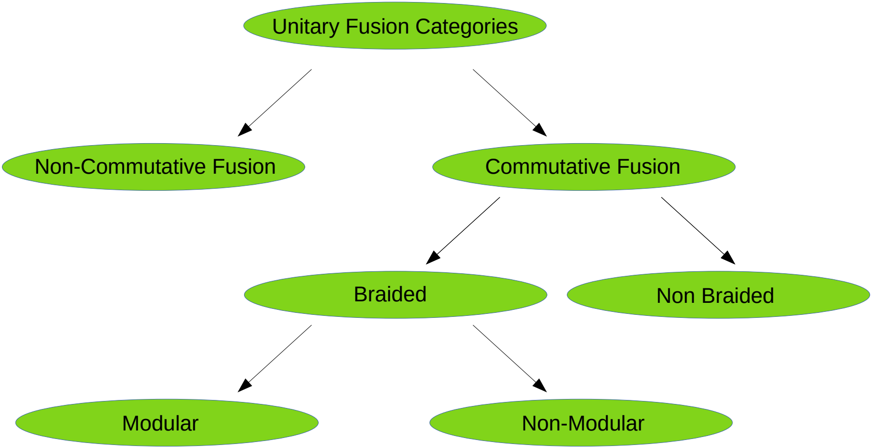
Output: Unitary Modular Tensor Category $\mathcal{Z}(\mathcal{C})$ (Drinfeld center of $\mathcal{C}$)

- Anyon theory (other objects, other fusion rules, other...)
- Achiral topological phase (chiral central charge $c \mod 8 = 0$)
- Different $\mathcal{C}$ may have the same $\mathcal{Z}(\mathcal{C})$ (Morita equivalence)

The Levin-Wen model in a nutshell

A possible tree of UFC



The Levin-Wen model in a nutshell

Local constraints and Hilbert space

- Degrees of freedom are the objects of $\mathcal{C}$
- Degrees of freedom are defined on the links of the trivalent graph
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

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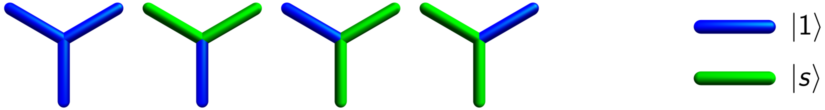
Ex: $\mathbb{Z}_2$ category

- Two labels: $\{1, s\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times s = s$, $s \times s = 1$



Hilbert space dimension for any trivalent graph with N_v trivalent vertices

- Dim $\mathcal{H} = 2^{\frac{N_\mathrm{v}}{2}+1}$

The Levin-Wen model in a nutshell

Local constraints and Hilbert space

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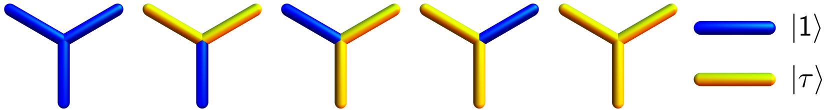
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

Ex: Fibonacci category

- Two objects: $\{1, \tau\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$



Hilbert space dimension for any trivalent graph with N_v vertices

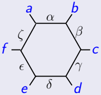
- $\dim \mathcal{H} = (1+\varphi^2)^{\frac{N_\text{v}}{2}} + (1+\varphi^{-2})^{\frac{N_\text{v}}{2}}, \quad \varphi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

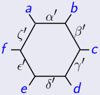
The Levin-Wen model in a nutshell

The Hamiltonian

$$H = -\sum_p P_p$$

- H is a sum of local commuting projectors: $[P_p, P_{p'}] = 0$
- P_p: projector onto the vacuum of $\mathcal{Z}(\mathcal{C})$ in the plaquette p

P_p 

$= \sum_s \frac{d_s}{D^2} \sum' F^{a\zeta\alpha}_{s\alpha'\zeta'} F^{b\alpha\beta}_{s\beta'\alpha'} F^{c\beta\gamma}_{s\gamma'\beta'} F^{d\gamma\delta}_{s\delta'\gamma'} F^{e\delta\epsilon}_{s\epsilon'\delta'} F^{f\epsilon\zeta}_{s\zeta'\epsilon'}$ 

- d_s: quantum dimension of the string $s \in \mathcal{C}$
- $D = \left(\sum_s d_s^2\right)^{1/2}$: total quantum dimension of $\mathcal{C}$

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)

Outline

Spectral degeneracies

On a genus-g orientable compact surface (sphere, torus,...)

- Ground-state degeneracy: $\mathcal{D}_0 = \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D} \right)^{\chi}$ =Turaev-Viro invariant
- D: total quantum dimension of $\mathcal{Z}(\mathcal{C})$
- Euler-Poincaré characteristic: $\chi = 2 - 2g$
- $g = 0$: $\mathcal{D}_0 = 1$
- $g = 1$: $\mathcal{D}_0 =$ Number of objects in $\mathcal{Z}(\mathcal{C})$
- $g \geqslant 2$: $\mathcal{D}_0$ depends (non trivially) on $\mathcal{Z}(\mathcal{C})$

Z. Kádár, A. Marzuoli, and M. Rasetti, Adv. Math. Phys. **2010**, 671039 (2010)

F. J. Burnell and S. H. Simon, Ann. Phys. **325**, 2550 (2010)

V. G. Turaev and O. Y. Viro, Topology **31**, 865 (1992)

G. Moore and N. Seiberg, Comm. Math. Phys. **123**, 177 (1989)

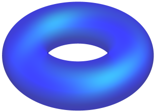
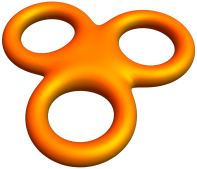
E. Verlinde, Nucl. Phys. B **300**, 360 (1988)

Spectral degeneracies

Ex: Fibonacci category (commutative, braided, modular)

- Objects of $\mathcal{C}$: $\{1, \tau\}$
- $d_1 = 1$, $d_\tau = \frac{1+\sqrt{5}}{2}$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{(1,1),(1,\tau),(\tau,1),(\tau,\tau)\}$
- $d_{(i,j)} = d_i \, d_j$

M. D. Schulz, S. Dusuel, K. P. Schmidt, and J. Vidal, Phys. Rev. Lett. **110**, 147203 (2013)
Y. Hu, S. D. Stirling, and Y.-S. Wu, Phys. Rev. B **85**, 075107 (2012)

$g = 0$	$g = 1$	$g = 2$	$g = 3$
			
$\mathcal{D}_0 = 1$	$\mathcal{D}_0 = 4$	$\mathcal{D}_0 = 25$	$\mathcal{D}_0 = 225$

Spectral degeneracies

Ex: Rep(S_3) category (commutative, braided, non-modular)

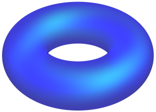
- Objects of $\mathcal{C}$: $\{1, 2, 3\}$
- $d_1 = 1,\ d_2 = 1,\ d_3 = 2$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{A, B, C, D, E, F, G, H\}$
- $d_A = 1,\ d_B = 1,\ d_C = 2,\ d_D = 3,\ d_E = 3,\ d_F = 2,\ d_G = 2,\ d_H = 2$

A. Kitaev, Ann. Phys. **303**, 2 (2003)
S. Beigi, P. W. Shor, D. Whalen, Commun. Math. Phys., **306**, 663 (2011)

$g = 0$

$\mathcal{D}_0 = 1$

$g = 1$

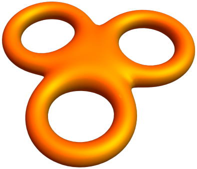


$\mathcal{D}_0 = 8$

$g = 2$

$\mathcal{D}_0 = 116$

$g = 3$



$\mathcal{D}_0 = 2948$

Spectral degeneracies

Ex: Tambara-Yamagami category TY_3 (commutative, non-braided)

- Objects of $\mathcal{C}$: $\{1, 2, 3, \sigma\}$
- $d_1 = d_2 = d_3 = 1,\ d_\sigma = \sqrt{3}$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{\alpha_{i=1,\ldots 15}\}$
- $d_{_{i=1,\ldots,6}} = 1,\ d_{_{i=7,\ldots,9}} = 2,\ d_{_{i=10,\ldots,15}} = \sqrt{3}$

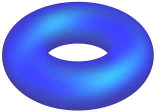
D. Tambara and S. Yamagami, J. Algebra **209**, 692 (1998)

S. Gelaki, D. Naidu, and D. Nikshych, Algebra Number Theory **3**, 959 (2009)

$g = 0$

$\mathcal{D}_0 = 1$

$g = 1$

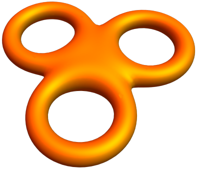


$\mathcal{D}_0 = 15$

$g = 2$

$\mathcal{D}_0 = 315$

$g = 3$



$\mathcal{D}_0 = 8\,883$

Spectral degeneracies

Ex: Haagerup subfactor category $\mathcal{H}_3$ (non-commutative)

- Objects of $\mathcal{C}$: $\{1, \alpha, \alpha^*, \rho, \rho\alpha, \rho\alpha^*\}$
- $d_1 = d_\alpha = d_{\alpha^*} = 1,\ d_\rho = d_{\rho\alpha} = d_{\rho\alpha^*} = \frac{3+\sqrt{13}}{2}$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{0, \mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \pi^1, \pi^2, \sigma^1, \sigma^2\, \sigma^3\}$
- $d_0 = 1,\ d_\mu = 3d_\rho,\ d_\pi = 3d_\rho + 1,\ d_\sigma = 3d_\rho + 2$

M. Asaeda and U. Haagerup, Comm. Math. Phys. **202**, 1 (1999)
S.-M. Hong, E. Rowell, and Z. Wang, Commun. Contemp. Math. **10**, 1049 (2008)

$g = 0$

$\mathcal{D}_0 = 1$

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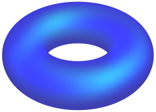
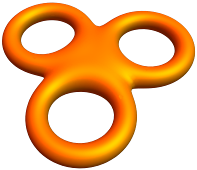


$\mathcal{D}_0 = 12$

$g = 2$

$\mathcal{D}_0 = 1401$

$g = 3$



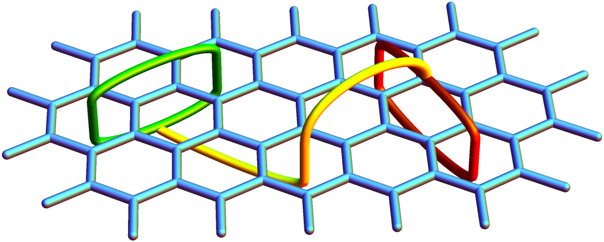
$\mathcal{D}_0 = 1\,603\,329$

Spectral degeneracies

Excitations of the Levin-Wen model (2005)

- Excitations are the nontrivial objects of $\mathcal{Z}(\mathcal{C})$
- General construction of the Drinfeld center via the Ocneanu tube algebra
- Some objects only violate the plaquette constraints (fluxons)
- Some objects violate the vertex/plaquette constraints (not considered here !)

T. Lan and X.-G. Wen, Phys. Rev. B **90**, 115119 (2014)



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Old results for commutative $\mathcal{C}$

- Number of states with q fluxons on a genus-g surface with N_p plaquettes:

$$\mathcal{D}_q = \binom{N_\mathrm{p}}{q} \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi - q} \left(1 - \frac{d_i}{D}\right)^q$$

G. Moore and N. Seiberg, Comm. Math. Phys. **123**, 177 (1989)

J. Vidal, Phys. Rev. B **105**, L041110 (2022)

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- n_i: internal multiplicity of the particle i given by the tube algebra

C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021)

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

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Example of Morita equivalence: Rep(S_3) and Vec(S_3)

- Same Drinfeld center $\mathcal{Z}(S_3)$: $\{A, B, C, D, E, F, G, H\}$
- Rep(S_3): $\mathbf{n} = (1, 0, 0, 1, 0, 1, 0, 0)$
- Vec(S_3): $\mathbf{n} = (1, 1, 2, 0, 0, 0, 0, 0)$
- Different Hilbert space, different degeneracies,...

Outline

Partition function

Energy spectrum

- Levin-Wen Hamiltonian: $H = -\sum_p P_p$
- Ground-state energy: $E_0 = -N_\mathrm{p}$
- $q-$fluxon state energy: $E_q = -N_\mathrm{p} + q$

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Exact finite-size and finite-temperature partition function

- Partition function: $Z = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \sum_{q=0}^{N_\mathrm{p}} \mathcal{D}_q \, \mathrm{e}^{-\beta E_q}$ $\left(\beta = \frac{1}{k_\mathrm{B} T}\right)$

$$\boxed{Z = \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi - N_\mathrm{p}} \left[n_i - \left(\frac{d_i}{D}\right)\left(1 - \mathrm{e}^{\beta}\right)\right]^{N_\mathrm{p}}}$$

- Z depends on the fusion rules: $d_a \times d_b = \sum_c N_c^{ab} d_c$
- Z depends on the surface topology: $\chi = 2 - 2g$
- Z depends on the number of plaquettes: N_p

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

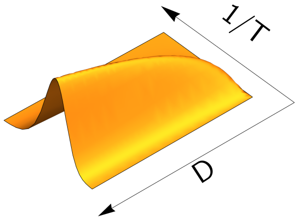
Partition function

Specific heat and absence of thermal phase transition

- Specific heat per plaquette : $c = \dfrac{\beta^2}{N_\mathrm{p}} \dfrac{\partial^2 \ln Z}{\partial \beta^2}$
- Thermodynamical limit ($N_\mathrm{p} \to \infty$): $c = \dfrac{\mathrm{e}^\beta \, \beta^2 (D-1)}{(D-1+\mathrm{e}^\beta)^2}$
- Only depends on the total quantum dimension D

No finite-temperature phase transition in the Levin-Wen model !

J. Vidal, Phys. Rev. B **105**, L041110 (2022)
A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation



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A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation

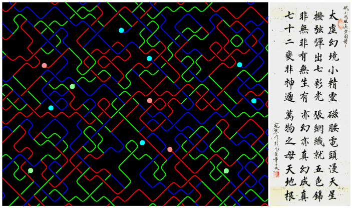
Absence of topological order at $T > 0$ for any <u>two-dimensional</u> Hamiltonian which is a sum of local commuting projectors !

M. Hastings, Phys. Rev. Lett. **107**, 210501 (2011)

Outlook

Summary and extensions

- Exact partition function of the Levin-Wen model
- More results available for:
  - any UFC with fusion multiplicities ($N_{ab}^{c} \geqslant 1$)
  - any topology with punctures (boundaries)
- Absence of finite-temperature phase transition
- Observables (e.g., Wegner-Wilson loops)
- Importance of charge excitations (violation of Gauss's law)



Courtesy of X.-G.Wen

What's next ?

Forthcoming conference in Les Houches (french Alps): 1-12 April 2024

Topological order: Anyons and Fractons

Website: https://topoanyons.sciencesconf.org/