Classifying Topological Many-Body Localized Phases

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University of Tübingen, 4 December 2023

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Bo Han (Weizmann)
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Sergii Strelchuk (Cambridge)
Amos Chan (Lancaster)
Joey Li (Innsbruck)
Steven Simon (Oxford)
Arijeet Pal (UCL)

European Commission

Marie Skłodowska-Curie Actions

Thermalization in classical systems

$\longrightarrow$

Many-body localization in one dimension

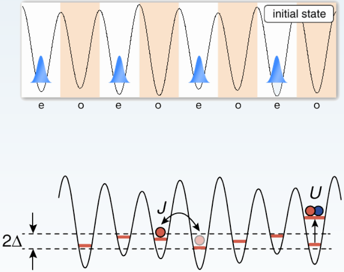
Sufficiently strong disorder in 1D $\Rightarrow$ ergodicity breaking:

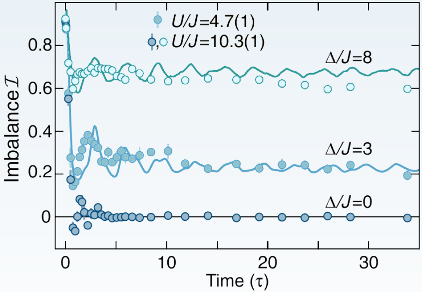
Many-body localization (MBL)

D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126 (2006).

I. Gornyi, A. Mirlin, and D. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).

Rigorous proof: J. Z. Imbrie, J. Stat. Phys. **163**, 998 (2016)





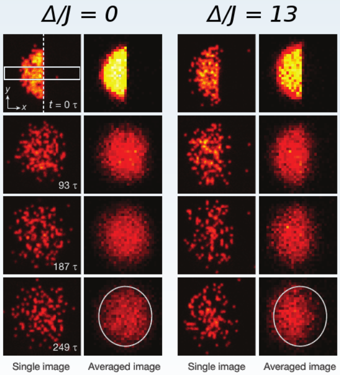
taken from: M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015)

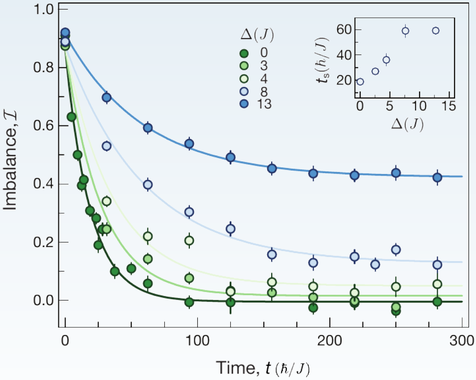
Many-body localization in higher dimensions?

Theralizing behavior in higher dimensions

W. De Roeck, J. Z. Imbrie, Phil. Trans. R. Soc. A **375**, 20160422 (2017).

But:





taken from: J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science **352**, 1547 (2016).

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Many-body localization (MBL)

Disordered Heisenberg antiferromagnet: MBL for $W > W_c \approx 3.5J$

$$H = \sum_{i=1}^{N} (J\,\mathbf{S}_i \cdot \mathbf{S}_{i+1} + h_i S_i^z), \qquad h_i \in [-W, W]$$

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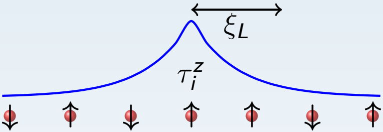


Local integrals of motion (LIOM):

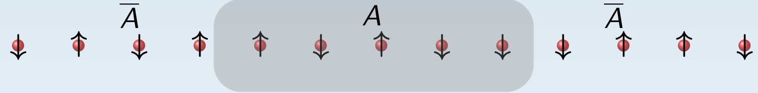
$$H = U H_{\mathrm{diag}} U^\dagger$$
$$\tau_i^z = U \sigma_i^z U^\dagger$$
$$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$$

M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013)

D. A. Huse and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014)

Area-law entangled eigenstates

$|\psi_n\rangle$:



$\rho_A = \mathrm{tr}_{\overline{A}}(|\psi_n\rangle\langle\psi_n|)$, entanglement entropy $S(\rho_A) \leq \mathrm{const.}$

M. Friesdorf, A. H. Werner, W. Brown, V. B. Scholz, and J. Eisert, Phys. Rev. Lett. **114**, 170505 (2015).

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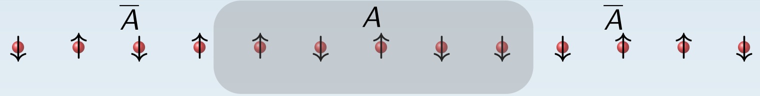

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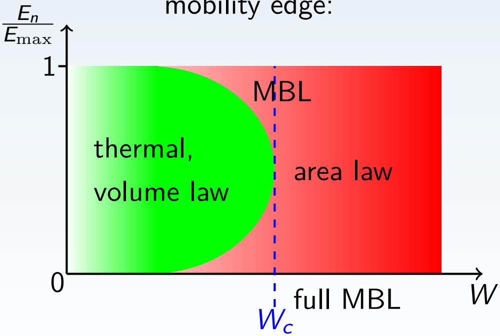
mobility edge:



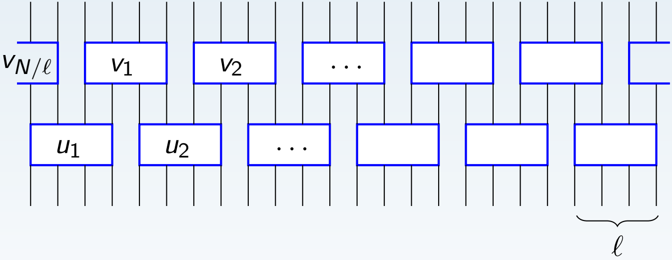
D. J. Luitz, N. Laflorencie, and F. Alet, Phys. Rev. B **91**, 081103 (2015)
However: W. De Roeck, F. Huveneers, M. Müller, and M. Schiulaz, Phys. Rev. B **93**, 014203 (2016)

Representation by Quantum Circuits

Goal

$\tilde{U} H \tilde{U}^\dagger \approx$ diagonal matrix

$\tilde{U} =$



$$\text{error} \sim e^{-\ell/\xi_L}$$

F. Pollmann, V. Khemani, J. I. Cirac, and S. L. Sondhi, Phys. Rev. B **94**, 041116 (2016),

T. B. Wahl, A. Pal, and S. H. Simon, Phys. Rev. X **7**, 021018 (2017)

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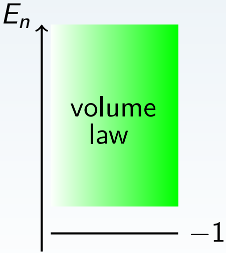
Topological Many-body Localized Phases

Cluster model:

Clean system

$$H = \sum_{j=1}^{N} \sigma^x_{j-1}\sigma^z_j\sigma^x_{j+1}$$

topological index: $ww^* = \pm\mathbb{1}$



Y. Bahri, R. Vosk, E. Altman and A. Vishwanath, Nat. Commun. **6**, 7341 (2015)
K. S. C. Decker, D. M. Kennes, J. Eisert, and C. Karrasch, Phys. Rev. B **101**, 014208 (2020)

Topological Many-body Localized Phases

Cluster model:

Clean system	Disordered system
$$H = \sum_{j=1}^{N} \sigma_{j-1}^{x} \sigma_{j}^{z} \sigma_{j+1}^{x}$$	$$H = \sum_{j=1}^{N} \lambda_j \sigma_{j-1}^{x} \sigma_{j}^{z} \sigma_{j+1}^{x}$$
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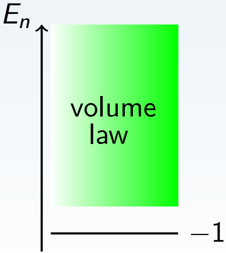

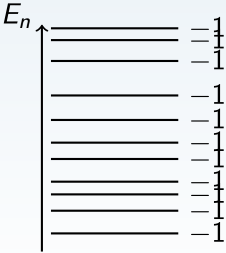

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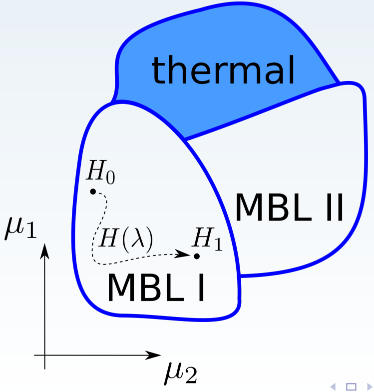
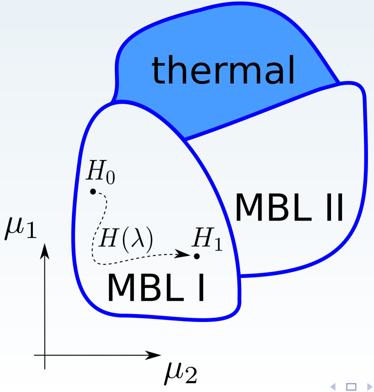
Topological MBL phase

Two MBL Hamiltonians H_0 and H_1 are in the same topological MBL phase iff one can continuously connect them via a path $H(\lambda)$ such that MBL is preserved along the path.



Symmetry-protected Topological MBL phase

Two MBL Hamiltonians H_0 and H_1 are in the same symmetry-protected topological MBL phase iff one can continuously connect them via a path $H(\lambda)$ such that MBL is preserved along the path, $H(\lambda) = u_g^{\otimes N} H(\lambda) u_g^{\dagger\otimes N}$.



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$\tilde{U} H \tilde{U}^\dagger \approx$ diagonal matrix

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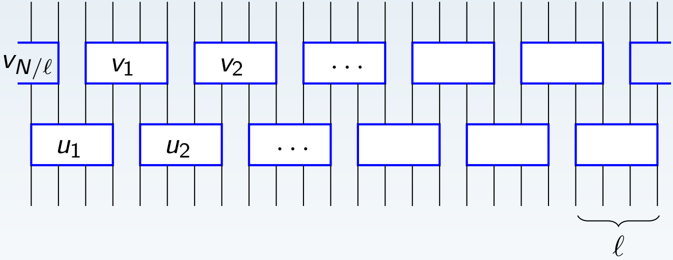

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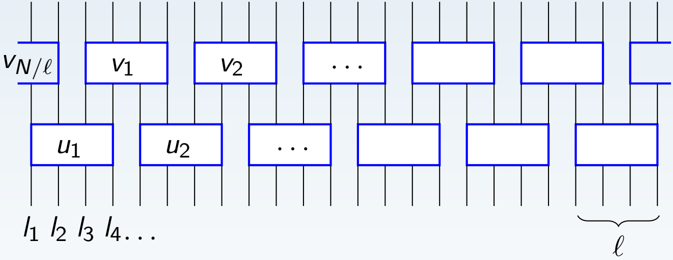

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MPS	Quantum Circuit
Ground states	All eigenstates
Translationally invariant, gapped	Disordered, many-body localized

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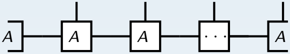

Bond dimension D

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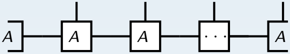
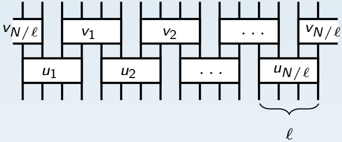
$\vert\tilde{\psi}\rangle =$ 	$\tilde{U} =$ 
Bond dimension D	Length of unitary gates ℓ ($D = 2^{\ell/2}$)

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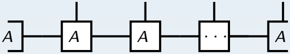
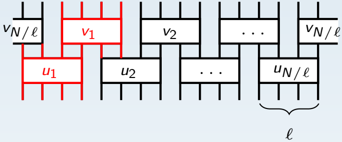
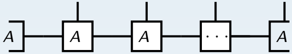
$\lvert\tilde{\psi}\rangle =$ 	$\tilde{U} =$ 
Bond dimension D	Length of unitary gates ℓ $(D = 2^{\ell/2})$

MPS

Ground states

Translationally invariant, gapped

$|\tilde{\psi}\rangle =$ 

Bond dimension D

Quantum Circuit

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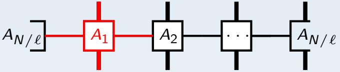
$\tilde{U} =$ 

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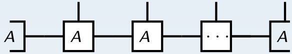
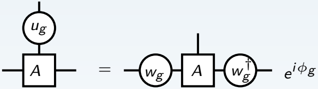
$|\tilde{\psi}\rangle =$ 

Bond dimension D

Symmetry $g \in G$: $H = u_g^{\otimes N} H \, u_g^{\dagger \otimes N}$



Quantum Circuit

All eigenstates

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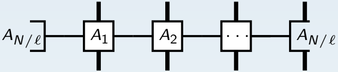
$\tilde{U} =$ 

Length of unitary gates ℓ ($D = 2^{\ell/2}$)

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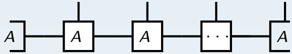

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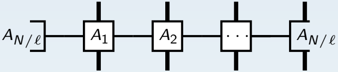
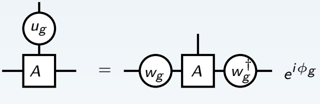
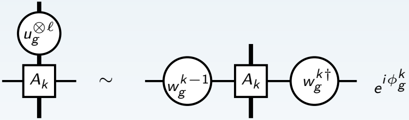


Bond dimension D	Length of unitary gates ℓ ($D = 2^{\ell/2}$)
Symmetry $g \in G$: $H = u_g^{\otimes N} H \, u_g^{\dagger \otimes N}$	Abelian symmetry group G : $H = u_g^{\otimes N} H \, u_g^{\dagger \otimes N}$





MPS

Ground states

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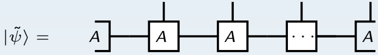

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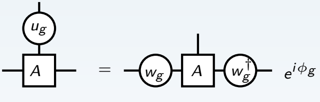


$$w_g w_h = w_{gh} e^{i\beta(g,h)}$$

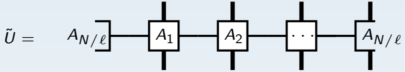
$$w'_g = w_g e^{i\chi_g} \;\Rightarrow\; \beta'(g,h) = \beta(g,h) - \chi_{gh} + \chi_g + \chi_h$$

$|\psi\rangle$: second cohomology group $H^2(G, U(1))$

Quantum Circuit

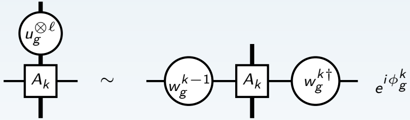
All eigenstates

Disordered, many-body localized



Length of unitary gates ℓ $(D = 2^{\ell/2})$

Abelian symmetry group G : $H = u_g^{\otimes N} H \, u_g^{\dagger \otimes N}$



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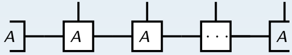
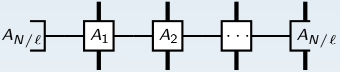
$\vert\tilde{\psi}\rangle =$ 	$\tilde{U} =$ 
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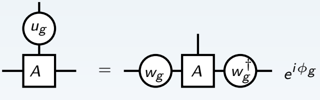
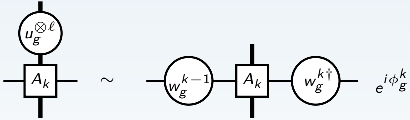
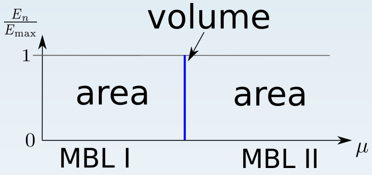
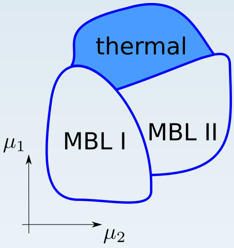
	
$w_g w_h = w_{gh} e^{i\beta(g,h)}$	$w_g^k w_h^k = w_{gh}^k e^{i\beta(g,h)}$
$w_g' = w_g e^{i\chi_g} \;\Rightarrow\; \beta'(g,h) = \beta(g,h) - \chi_{gh} + \chi_g + \chi_h$	$w_g^{k\prime} = w_g^k e^{i\chi_g} \;\Rightarrow\; \beta'(g,h) = \beta(g,h) - \chi_{gh} + \chi_g + \chi_h$
$\vert\psi\rangle$: second cohomology group $H^2(G, U(1))$	$\tilde{U} \ni \vert\tilde{\psi}_{l_1 \ldots l_N}\rangle$: second cohomology group $H^2(G, U(1))$
Schuch, Pérez-García, and Cirac, PRB **84**, 165139 (2011)	Wahl, PRB **98**, 054204 (2018), Chan and Wahl, JPCM **32**, 305601 (2020)

MBL topologial phase transition

Scenario 1:





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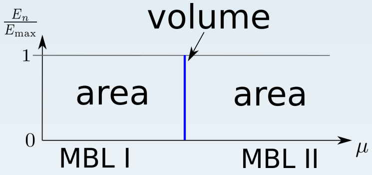
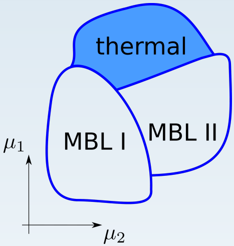
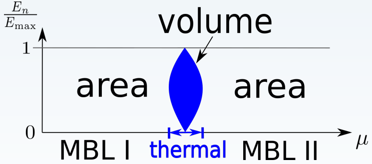
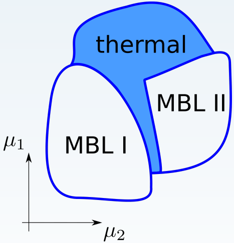




Scenario 2:

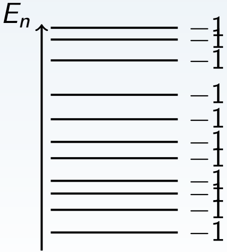


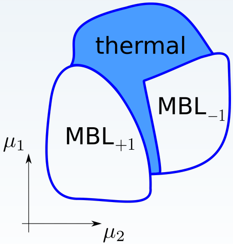


S. Moudgalya, D. A. Huse, and V. Khemani, arXiv:2008.09113.
R. Sahay, F. Machado, B. Ye, C. R. Laumann, and N. Y. Yao, Phys. Rev. Lett. **126** (2021).
T. B. Wahl, F. Venn, and B. Béri, Phys. Rev. B **105**, 144205 (2022).

Intermediate Summary

- In MBL systems, all eigenstates are area-law entangled
- For symmetry-protected topological MBL, all eigenstates must have the same topological label
- MBL gets destroyed when the topological label changes
  $\Rightarrow$ different topological labels correspond to different symmetry-protected topological MBL phases





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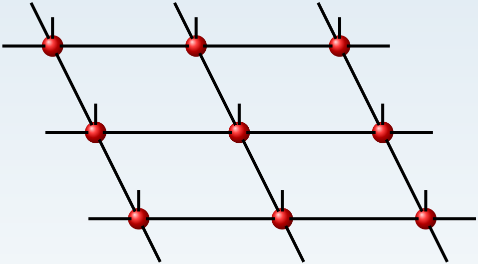
In two dimensions

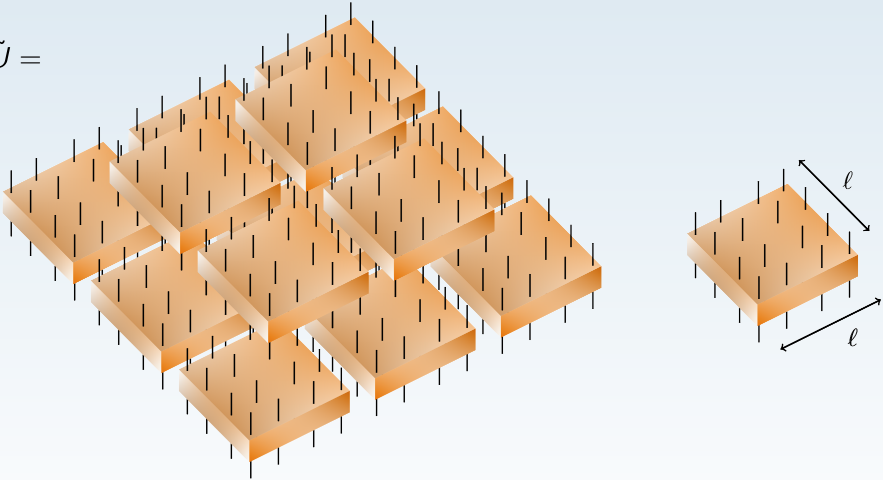
$$|\tilde{\psi}\rangle =$$



becomes ...

In two dimensions

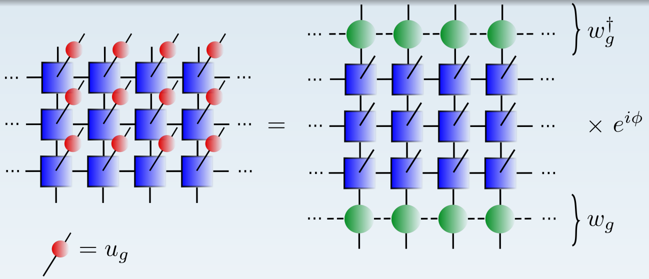
$\tilde{U} =$



F. Pollmann, V. Khemani, J. I. Cirac, and S. L. Sondhi, Phys. Rev. B **94**, 041116 (2016),

T. B. Wahl, A. Pal, and S. H. Simon, Nat. Phys. **15**, 164 (2019)

Classification of 2D symmetry-protected MBL phases



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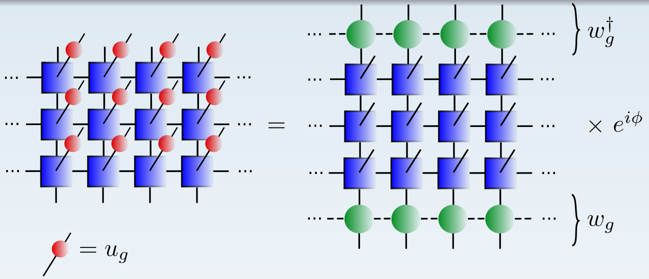

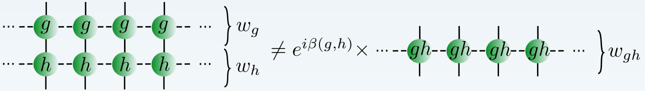

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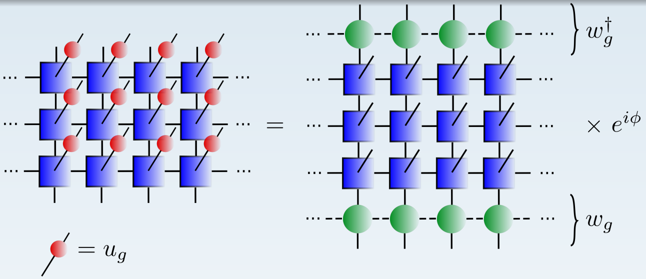
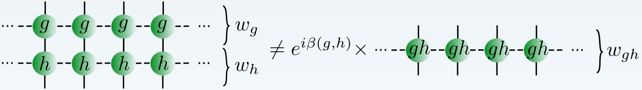
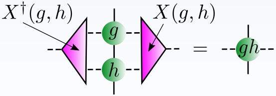




w_g injective $\Rightarrow$



classify complex phases of $X(g,h) \Rightarrow H^3(G, U(1))$

D. J. Williamson, N. Bultinck, M. Mariën, M. B. Şahinoğlu, J. Haegeman, and F. Verstraete, Phys. Rev. B **94**, 205150 (2016),

J. Li, A. Chan, and T. B. Wahl, Phys. Rev. B **102**, 014205 (2020)

Classification Table

Classification using **non-translationally invariant** quantum circuits:

Symmetry (spin systems)	Topological classes
1D, time-reversal symmetry	$\mathbb{Z}_2$
1D, on-site symmetry G	$\mathcal{H}^2(G, U(1))$
2D, time-reversal symmetry	$\{0\}$
2D, on-site symmetry G	$\mathcal{H}^3(G, U(1))$

T. B. Wahl, Phys. Rev. B **98**, 054204 (2018).
A. Chan and T. B. Wahl, J. Phys.: Cond. Mat. **32**, 305601 (2020).
J. Li, A. Chan, and T. B. Wahl, Phys. Rev. B **102**, 014205 (2020).

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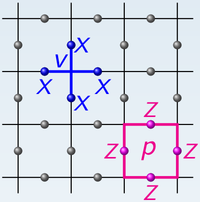
Topologically ordered ground states

Example: **Toric code**

$$H = -\sum_v A_v - \sum_p B_p$$

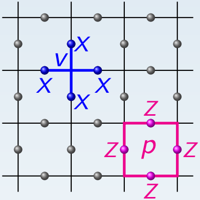
$$[H, A_v] = [H, B_p] = [A_v, B_p] = 0$$



- four ground states on the torus: $|\psi_j\rangle,\ j = 1, 2, 3, 4$
- cannot be connected to product state via local unitary $U_{\rm loc}$: $|\psi_{\rm prod}\rangle \neq U_{\rm loc}|\psi_j\rangle$

Topologically ordered many-body localization

Example: **Random coupling toric code**



$$H = -\sum_v J_v A_v - \sum_p K_p B_p$$

$$[H, A_v] = [H, B_p] = [A_v, B_p] = 0$$

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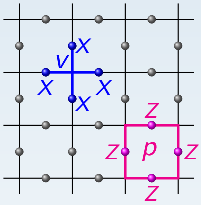


~~Local~~ integrals of motion:

$$H = U H_{\text{diag}} U^\dagger$$

$$\tau_i^z = U \sigma_i^z U^\dagger$$

$$[H, \tau_i^z] = [\tau_i^z, \tau_j^z] = 0$$

Alternative choice:

$$S_i = A_v, B_p$$

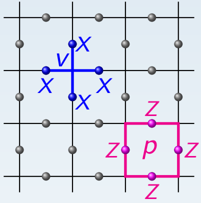
$$\Rightarrow [H, S_i] = [S_i, S_j] = 0$$

Topologically ordered many-body localization

Example: **Random coupling toric code + perturbation**

$$H = -\sum_v J_v A_v - \sum_p K_p B_p + \sum_i h_i \sigma_i^z$$

$$[H, A_v] = [H, B_p] = [A_v, B_p] = 0$$



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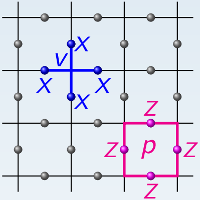


Topological local integrals of motion:

stabilizers S_i (Abelian, non-chiral)

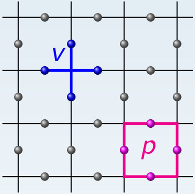
$$T_i = U_{\mathrm{loc}} S_i U_{\mathrm{loc}}^\dagger$$

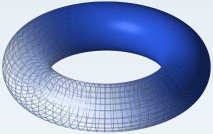
$$[H, T_i] = [T_i, T_j] = 0$$

T. B. Wahl and B. Béri, Phys. Rev. Res. **2**, 033099 (2020).
T. B. Wahl, F. Venn, and Béri, arXiv:2111.11543.

1. All eigenstates are in same topological phase
2. Stable unless perturbations are strong enough to destroy MBL

Topologically ordered MBL in 2D: Numerical simulations





A. Kitaev, Ann. Phys. 303, **2** (2003)

$$H = -\sum_{v} J_v A_v - \sum_{p} K_p B_p + \sum_{i} h_i \sigma_i^z$$

with $J_v, K_p \sim \mathcal{N}(0,1),\; h_i \sim \mathcal{N}(0,\sigma)$

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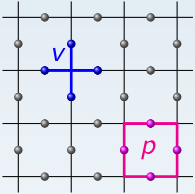

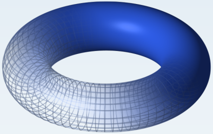

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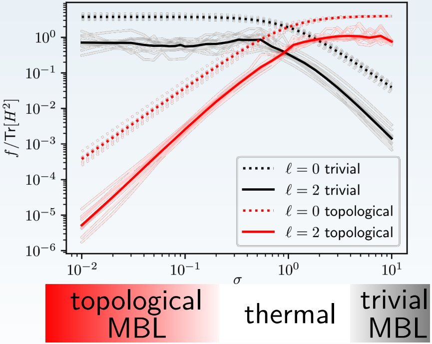
$T_i = U_\ell S_i U_\ell^\dagger$:

(i) $S_i = A_v, B_p$

(ii) $S_i = \sigma_i^z$

10×10 lattice; error function:



F. Venn, T. B. Wahl, and B. Béri, arXiv:2212.09775

Summary and Outlook

Summary:

- **SPT:** classification by second cohomology group in 1D, classification by third cohomology group in 2D
- **Top. Order:** extended definition of local integrals of motion
- All eigenstates are in the same topological phase
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Summary and Outlook

Summary:

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Outlook:

- Are there topological features invisible in individual eigenstates (only encoded in the overall unitary)?
- What is the nature of the thermal phase separating topologically distinct MBL phases?

Thorsten B. Wahl, Phys. Rev. B **98**, 054204 (2018).
Amos Chan and Thorsten B. Wahl, J. Phys.: Cond. Mat. **32**, 305601 (2020).
Joey Li, Amos Chan, and Thorsten B. Wahl, Phys. Rev. B **102**, 014205 (2020).
Thorsten B. Wahl and Benjamin Béri, Phys. Rev. Research **2**, 03309 (2020).
Thorsten B. Wahl, Florian Venn, and Benjamin Béri, Phys. Rev. B **105**, 144205 (2022).
Florian Venn, Thorsten B. Wahl, and Benjamin Béri, arXiv:2212.09775.