Entanglement of Gauge Theories: from the Toric Code to the Z2 Lattice Gauge Higgs Mode

Wen-Tao Xu (TUM, Munich)

In collaboration with Michael Knap and Frank Pollmann

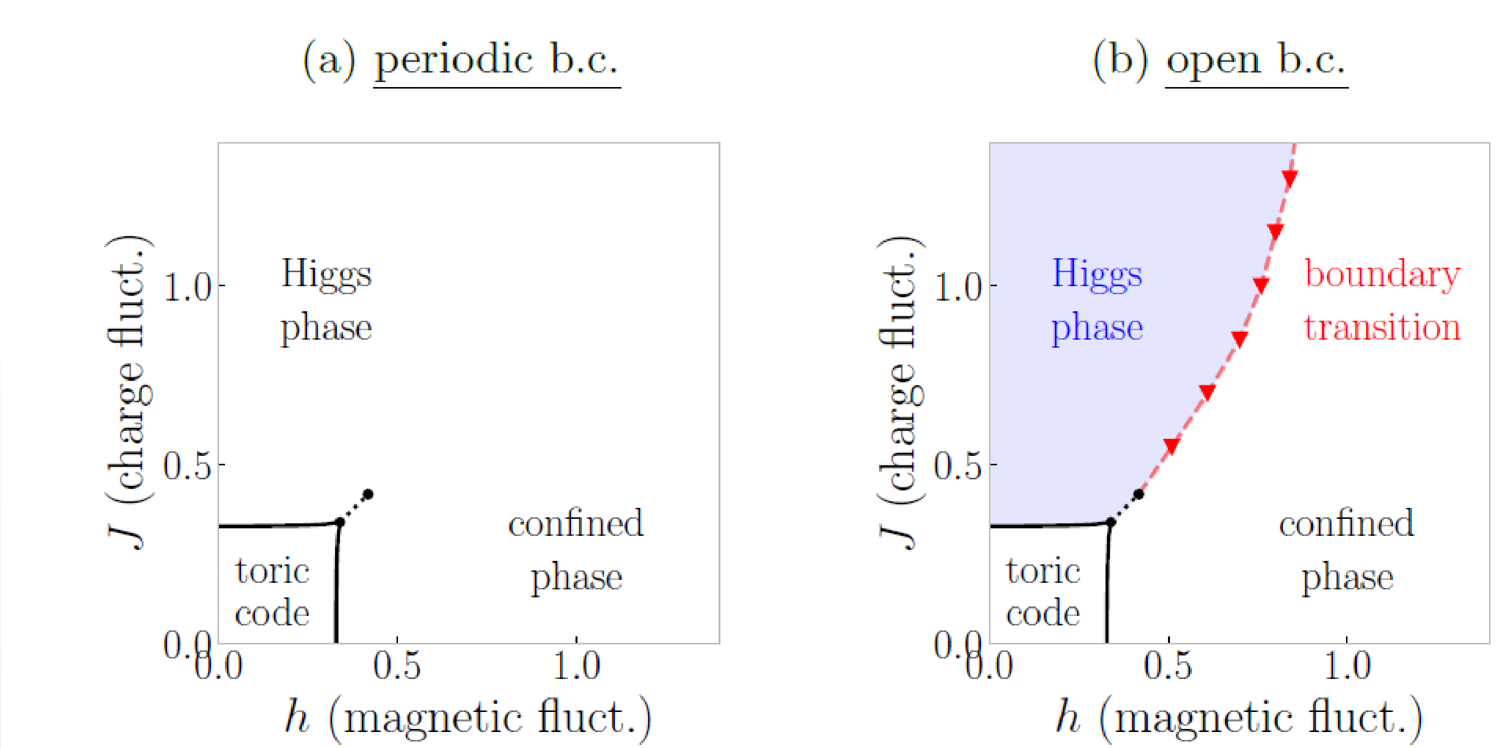
November 29th, 2023, Tübingen

Reference: arXiv: 2311.16235

Entanglement of the $\mathbb{Z}_2$ gauge Higgs model

- Entanglement of gauge theories is largely unexplored
- A boundary phase transition, can we observe in entanglement spectrum

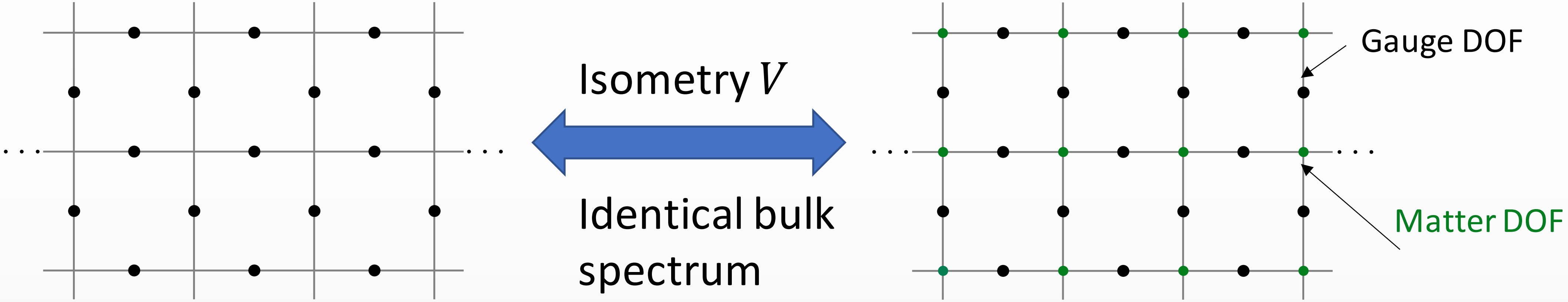
Kogut, RMP, (1979)
G. A. Jongeward, PRD **21**, 3360 (1980)
I.S. Tupitsyn et.al, PRB (2010)



R. Verresen, et.al, arXiv.2211.01376

Toric code mode and the $\mathbb{Z}_2$ gauge Higgs model

Toric code model: H_{TC} 　　　 $\mathbb{Z}_2$ gauge Higgs model: H_{GH}



I.S. Tupitsyn et.al, PRB (2010)

- Relation between Hamiltonians: $H_{GH} = V H_{TC} V^\dagger$
- Q: *what is the relation between their entanglement structures*?
- $|\Psi_{GH}\rangle = V|\Psi_{TC}\rangle \longrightarrow \rho_{GH} \overset{\text{Quantum channel}}{\longleftrightarrow} \rho_{TC}$

Outline

- Toric code (TC) model (in a magnetic field) and $\mathbb{Z}_2$ gauge Higgs (GH) model
- Quantum channel mapping between reduce density matrices of two models
- Contrast ground state entanglement Hamiltonian (EH) and entanglement spectra (ES) of TC and GH model
- Investigate the entanglement distillation of the $\mathbb{Z}_2$ GH model.

Toric code model in a magnetic field

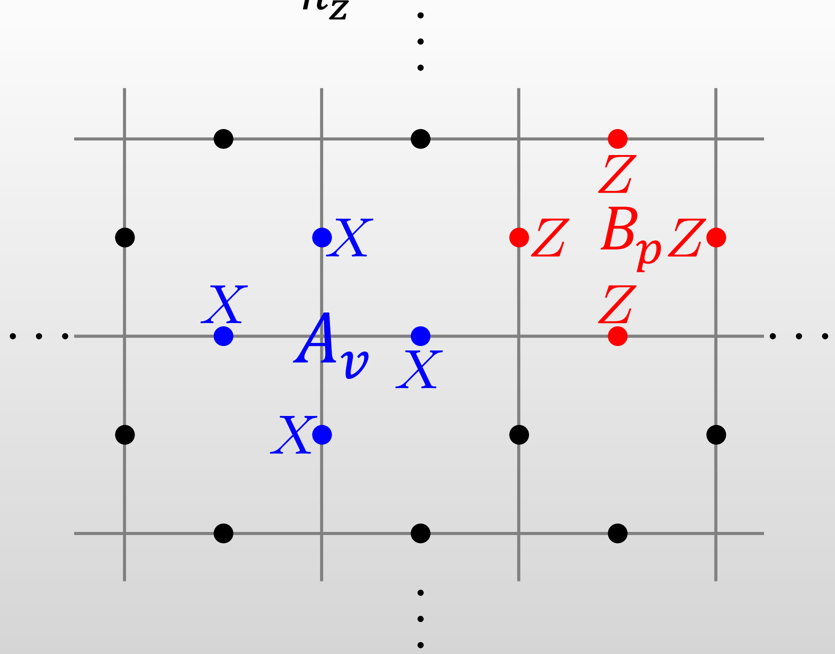
- Toric code model in a magnetic field: Kitaev, Ann. Phys. (2003)

$$H_{TC} = -\sum_v A_v - \sum_p B_p - h_x \sum_e X_e - h_z \sum_e Z_e , \qquad A_v = \prod_{e\in v} X_e , B_p = \prod_{e\in p} Z_e ,$$

- Polar coordinate parameterization: $h = \sqrt{h_x^2 + h_z^2}, \theta = \arctan\frac{h_x}{h_z}$
- Electric-magnetic (EM) duality:

U_{TC}: $X \leftrightarrow Z$, primal lattice $\leftrightarrow$ dual lattice

$$U_{TC} H_{TC}(h_x, h_z) U_{TC}^\dagger = H_{TC}(h_z, h_x)$$



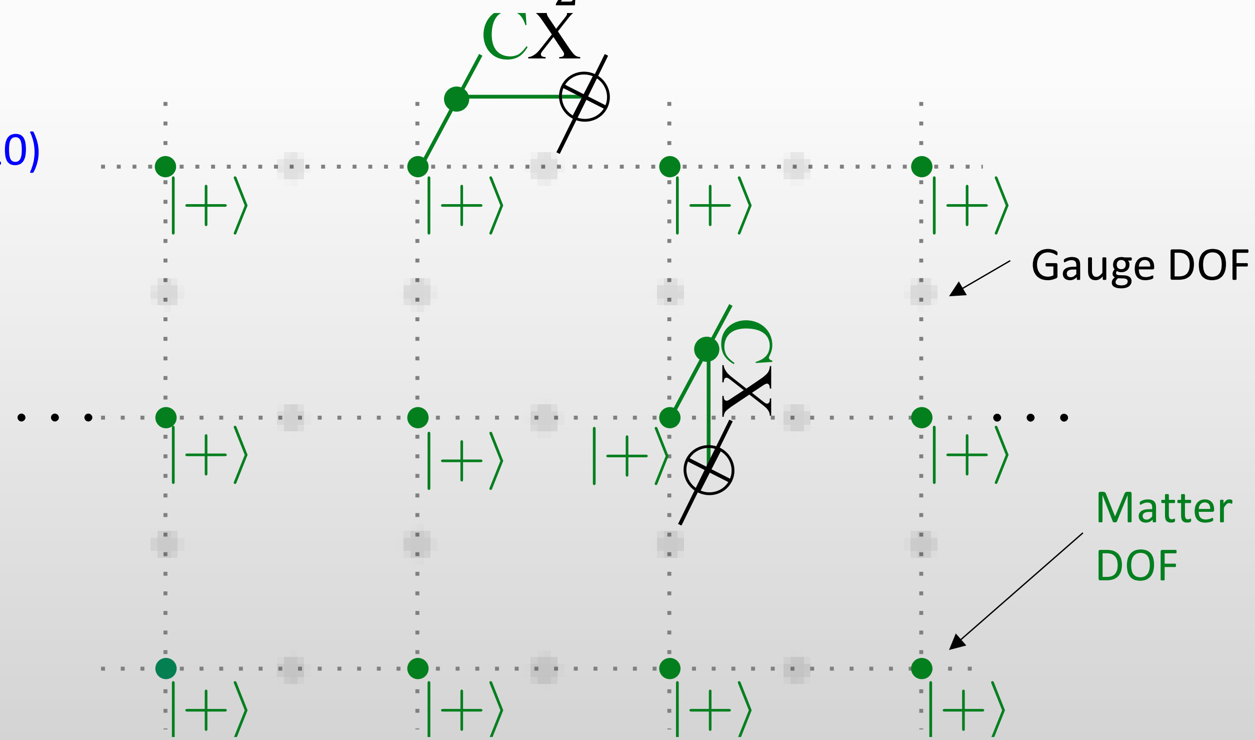
Isometry transformation

- An isometry transformation from toric code model to $\mathbb{Z}_2$ gauge Higgs model:

$$V = \prod_{\langle v,e \rangle} CX_{v,e} \prod_{v} |+\rangle_v$$

- CX: control X; $V^\dagger V = \mathbb{I}$, $VV^\dagger = \prod_v \frac{1+X_v A_v}{2}$.

I.S. Tupitsyn et.al, PRB (2010)



$\mathbb{Z}_2$ gauge Higgs model

- $\mathbb{Z}_2$ gauge Higgs model: Kogut and Susskind, PRD, 1975

$$H_{GH} = VH_{TC}V^\dagger = -\sum_v X_v - \sum_p B_p - h_x \sum_e X_e - h_z \sum_e Z_{v(e)} Z_e Z_{v'(e)},$$

where

$$A_v = \prod_{e \in v} X_e \, , B_p = \prod_{e \in p} Z_e$$

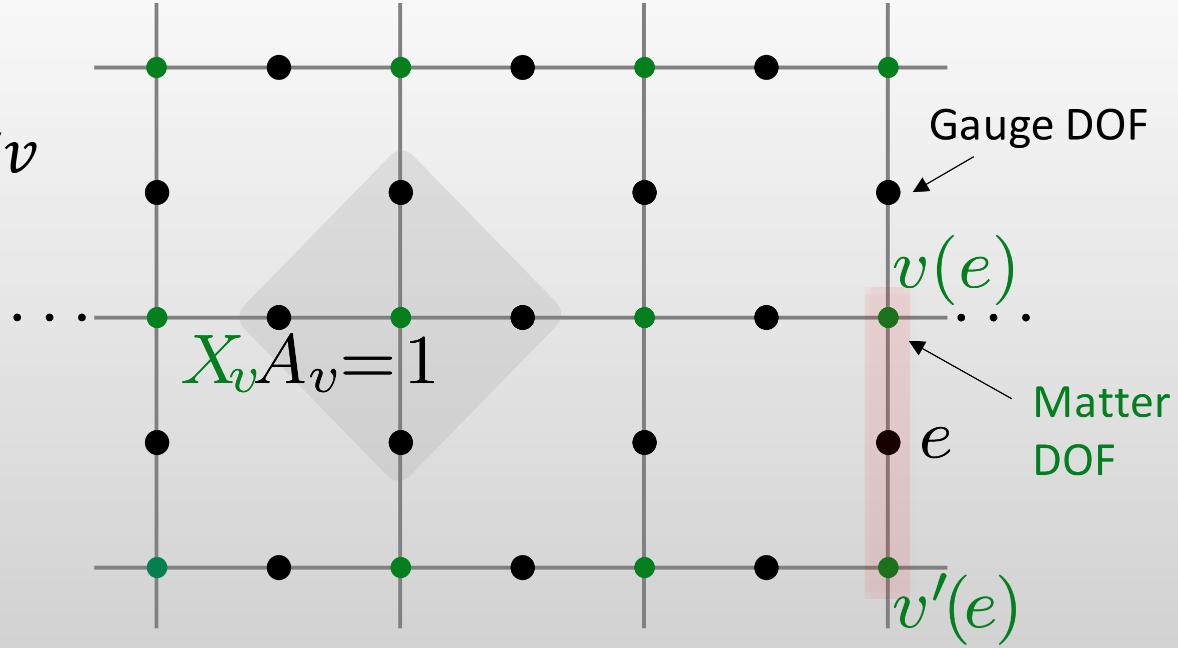
- h_x gauge fluctuation, h_z gauge-matter coupling.

- Gauge constraint (Gauss law): $X_v A_v = 1 \; \forall v$

  from the isometry $VV^\dagger = \prod_v \frac{1+X_v A_v}{2}$

- $h_z = 0$, pure $\mathbb{Z}_2$ lattice gauge theory:

- $H_{GH} = -\sum_v A_v - \sum_p B_p - h_x \sum_e X_e$



Phase diagrams and bipartition

- Same phase diagram: $H_{GH} = V H_{TC} V^\dagger$
- EM duality U_{TC} for TC:

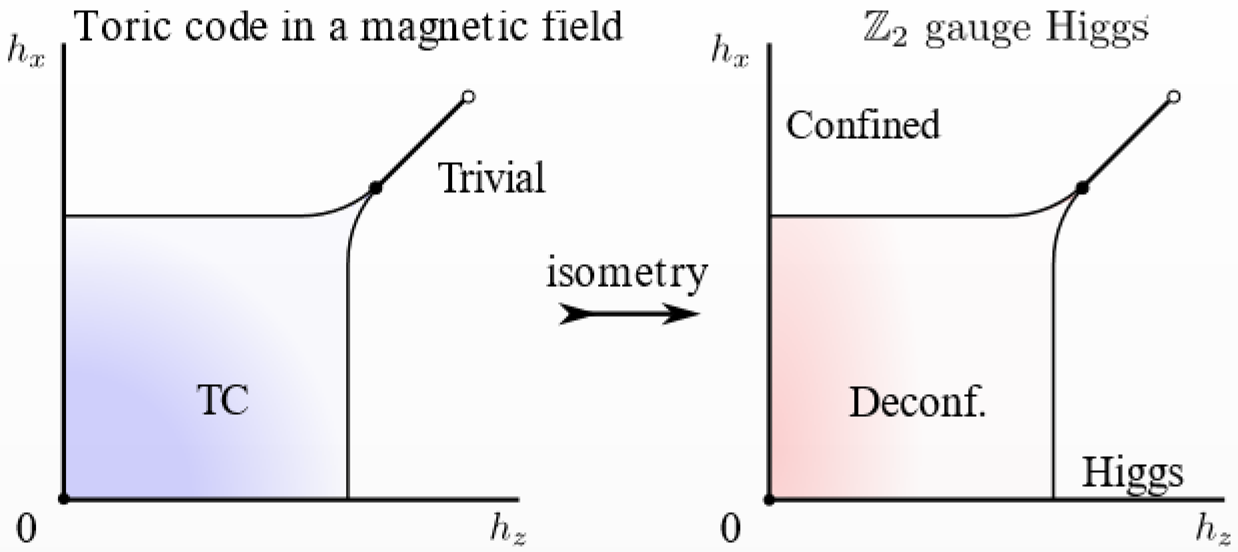
$$U_{TC} H_{TC}(h_x, h_z) U_{TC}^\dagger = H_{TC}(h_z, h_x)$$

- EM duality for $\mathbb{Z}_2$ GH: $U_{GH} = V U_{TC} V^\dagger$

$$U_{GH} H_{GH}(h_x, h_z) U_{GH}^\dagger = H_{GH}(h_z, h_x)$$

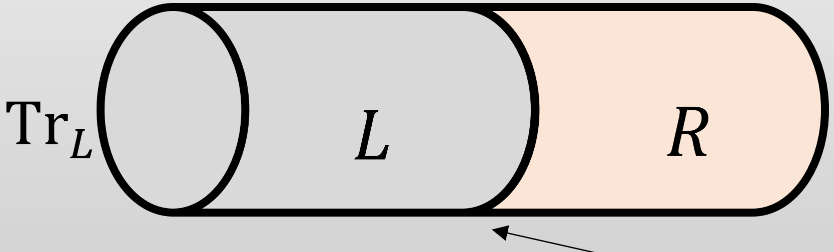
- Entanglement bipartition:

$$\rho_{TC} = \mathrm{Tr}_L |\Psi_{TC}\rangle\langle\Psi_{TC}|$$
$$\rho_{GH} = \mathrm{Tr}_L |\Psi_{GH}\rangle\langle\Psi_{GH}|$$

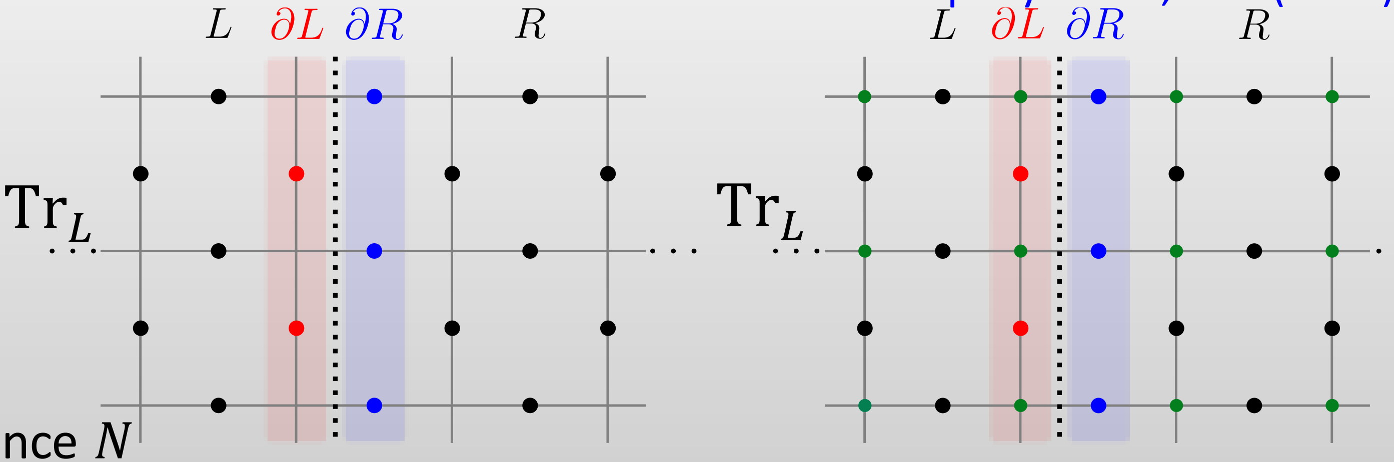


F. Wu, et.al PRB.85.195104 (2012)
J. Vidal, PRB.79.033109 (2009)

Kogut, RMP, (1979)
G. A. Jongeward, PRD **21**, 3360 (1980)
I.S. Tupitsyn et.al, PRB (2010)

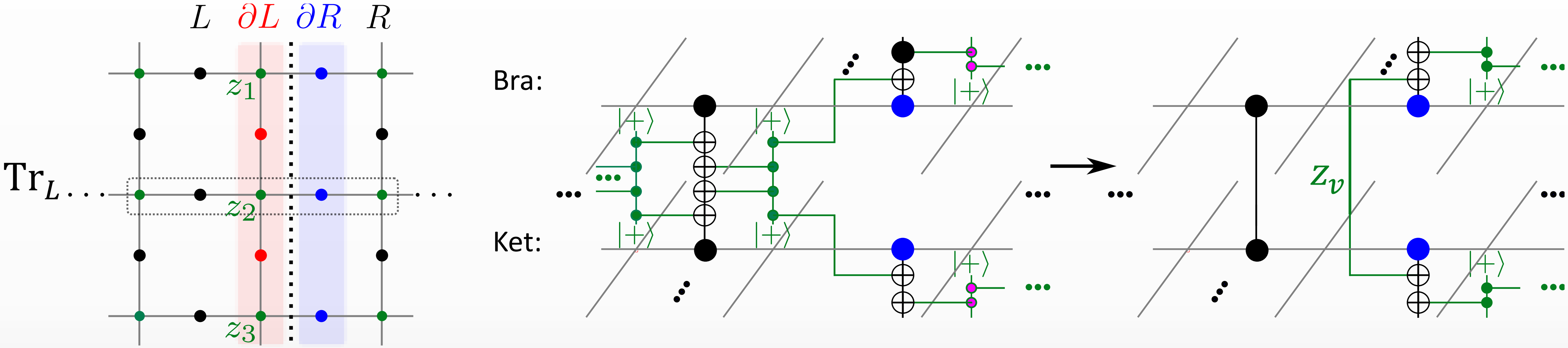


circumference N



Quantum channel

- Since $|\Psi_{GH}\rangle = V|\Psi_{TC}\rangle$, what is the relation between ρ_{TC} and ρ_{GH}?



$$\mathrm{Tr}_L V|\Psi_{TC}\rangle\langle\Psi_{TC}|V^\dagger$$

- ρ_{TC} and ρ_{GH} are related by a quantum channel $\mathcal{N}$:

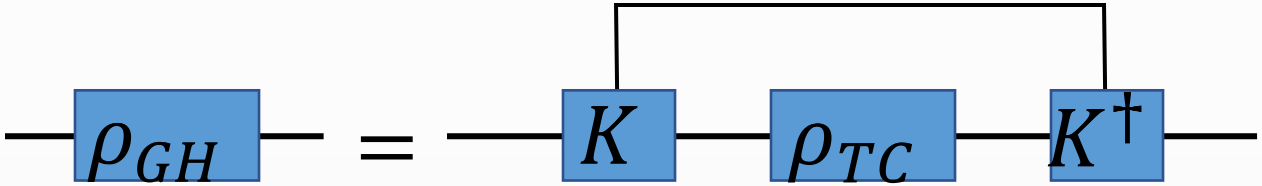
$$\rho_{GH} = \mathcal{N}(\rho_{TC}) = \sum_{z_1\cdots z_N} K_{z_1\cdots z_N}\rho_{TC}K^\dagger_{z_1\cdots z_N}$$

- Kraus operators: $K_{z_1\cdots z_N} = \frac{1}{2^N}\prod_{\langle e\in R, v\in R\cup\partial L\rangle} CX_{v,e}\prod_{v\in\partial L}|z_v\rangle_v \otimes \prod_{v\in R}|+\rangle_v$

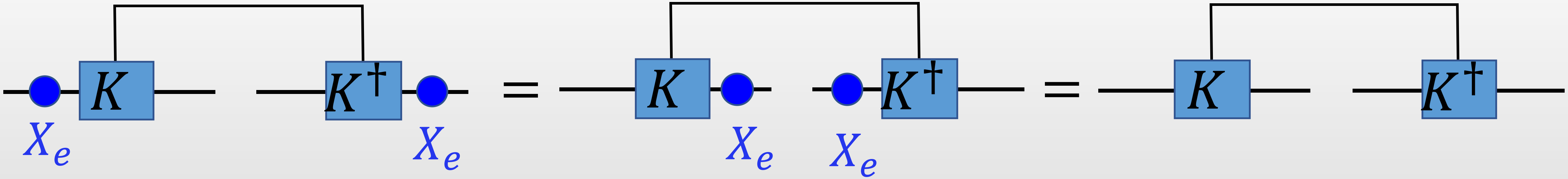
Quantum channel

- Graphic notation

$$\rho_{GH} = \mathcal{N}(\rho_{TC}) = \sum_{z_1 \cdots z_N} K_{z_1 \cdots z_N} \rho_{TC} K^{\dagger}_{z_1 \cdots z_N}$$



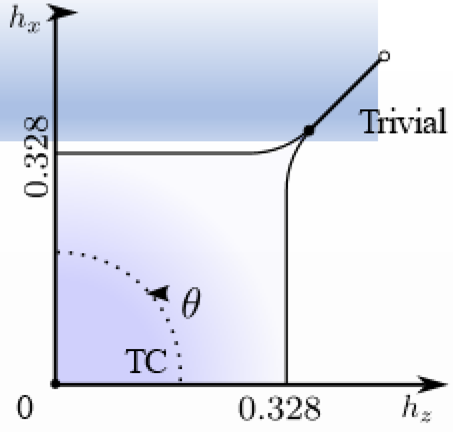
- Gauge symmetry of the quantum channel ($\forall\ e \in \partial R$)



- Obtain all entanglement properties of $\mathbb{Z}_2$ GH model via solution of the TC model

- Find how symmetry applies on ρ_{GH}, which might be unusual.

Ground state EH of TC

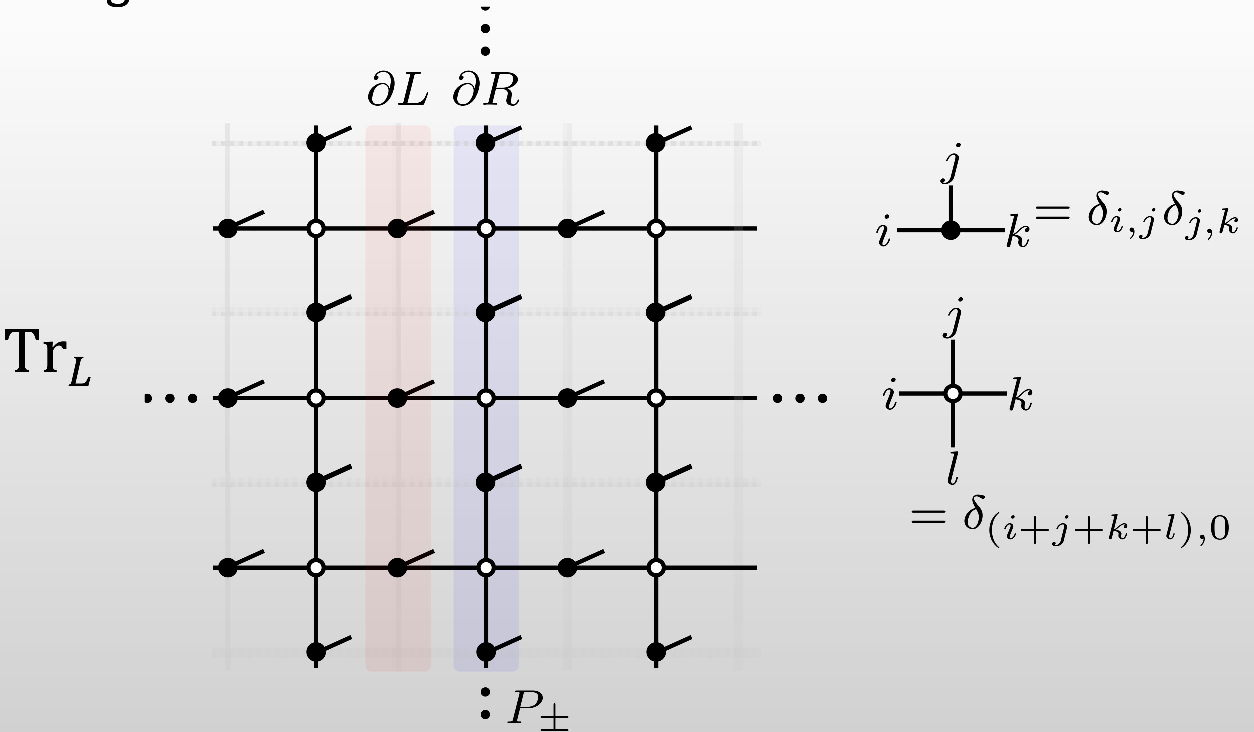


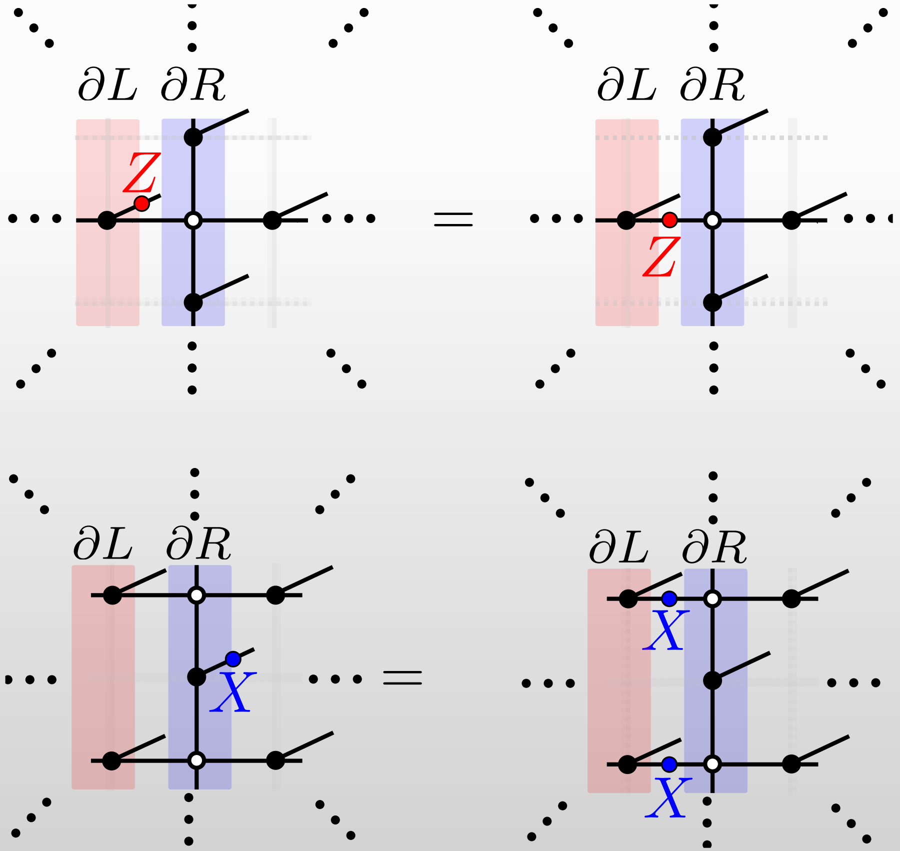
- Perturbed ground state wavefunction:

$$|\Psi(h_x, h_z)\rangle = \sum_e \left[1 + \frac{h_x}{4} X_e + \frac{h_z}{4} Z_e + O(h^2)\right] |\mathrm{TC}\rangle,$$

where $|\mathrm{TC}\rangle = |\Psi(0,0)\rangle$. Only $\sum_{e\in\partial R} X_e$ and $\sum_{e\in\partial L} Z_e$ contribute the EH:

Single line PEPS:





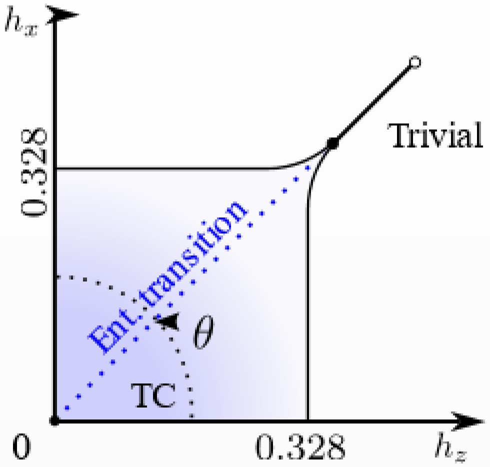
Ground state EH & ES of TC

W. W. Ho et.al.,, PRB.91.125119(2015)

- EH of a ground state with trivial anyon flux

$$H_{E,TC} = P_+ \left[\log 2^{N-1} - \sum_{i=1}^{N} \left(\frac{h\cos\theta}{2} Z_i + \frac{h\sin\theta}{2} X_i X_{i+1} \right) + O(h^2) \right]$$

where $P_+ = (1 + \prod_i Z_i)/2$.

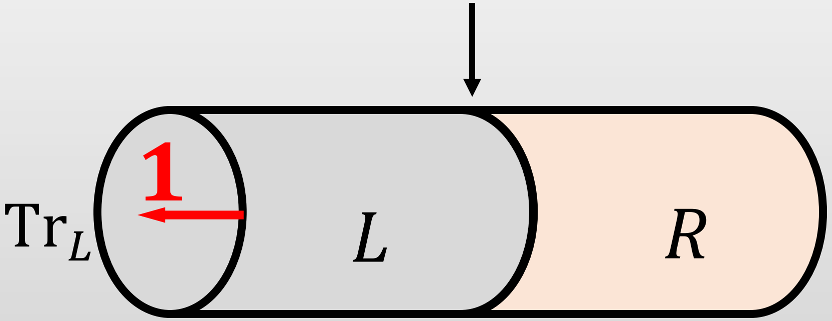


- ES from PEPS; $\{\lambda_i\} = \text{eig}\left(\rho_{TC} \sim \sqrt{\sigma_L^T}\sigma_R\sqrt{\sigma_L^T}\right)$, $\Delta_i = \frac{-(\log\lambda_i - \log\lambda_0)}{-\log\lambda_1}$

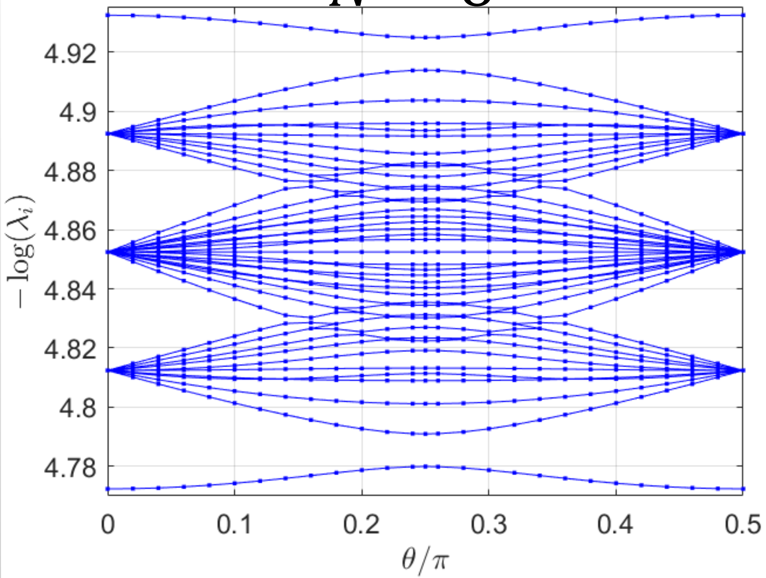
where σ_L and σ_R are left and right PEPS transfer matrix fixed points

J. I. Cirac, et.al., PRB.83.245134 (2011)

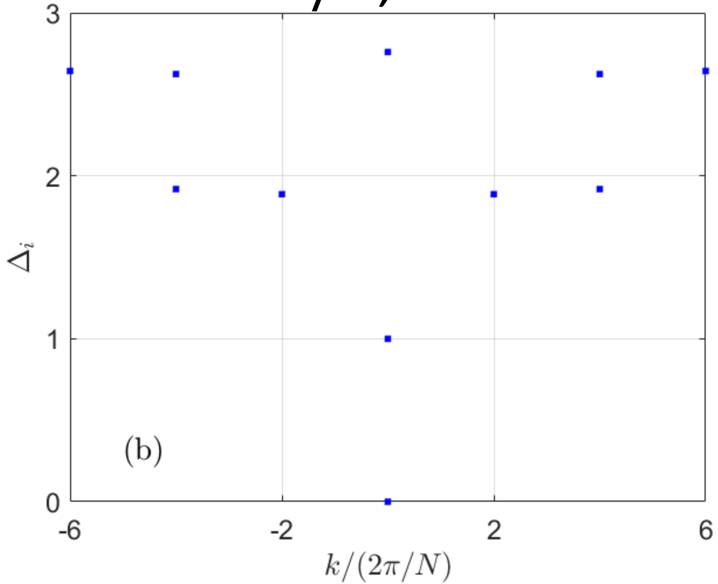
Circumference N



$N = 8$

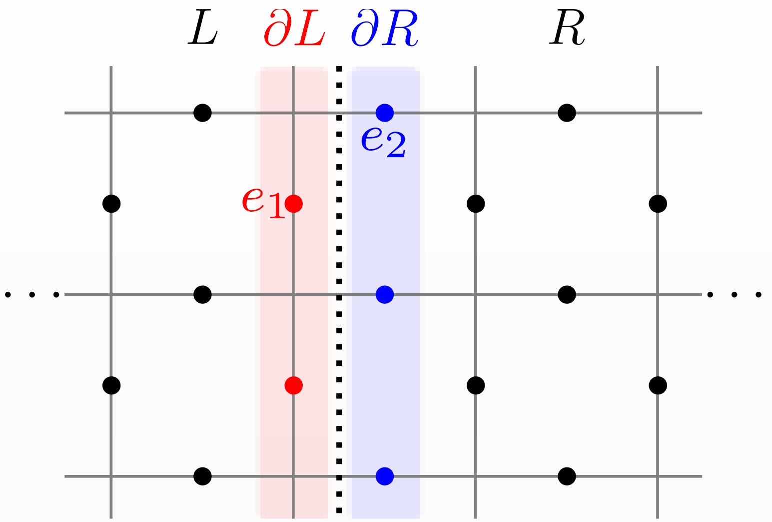


$\theta = \pi/4, N = 10$

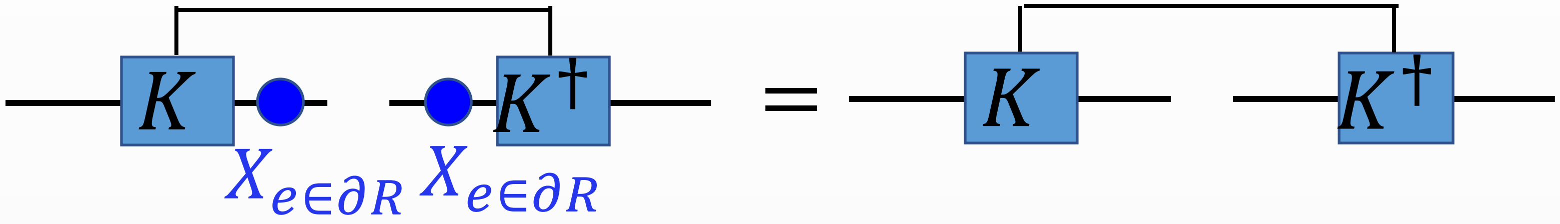


Ground state EH of $\mathbb{Z}_2$ GH model

- Obtain ES and EH of $\mathbb{Z}_2$ GH model from $\rho_{GH} = \mathcal{N}(\rho_{TC})$.

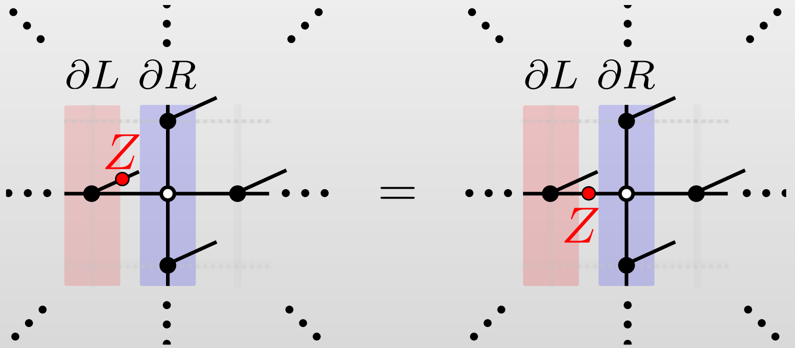


- If $\exists e \in \partial R$ s.t. $\{O, X_e\} = 0$, $\mathcal{N}(O) = 0$.



- $\rho_{TC} = |\mathrm{TC}\rangle\langle\mathrm{TC}| + \sum_e \left(\frac{h_x}{4} X_e + \frac{h_z}{4} Z_e\right) |\mathrm{TC}\rangle\langle\mathrm{TC}| + h.c. + O(h^2)$
- $\{Z_{e_1} |\mathrm{TC}\rangle\langle\mathrm{TC}|, X_{e_2}\} = 0$, so no term $\sum_i Z_i$ in EH of $\mathbb{Z}_2$ GH model

- EH: a classical Ising chain

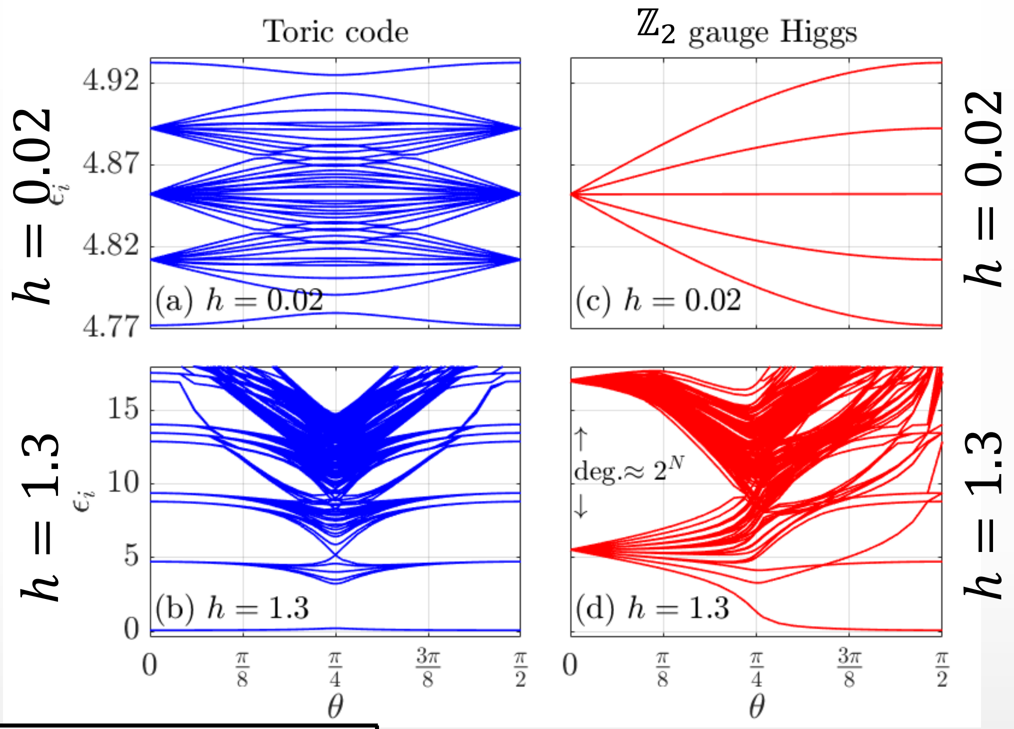


$$H_{E,GH} = P_+ \left[\log 2^{N-1} - \frac{h_x}{2} \sum_{i=1}^{N} X_i X_{i+1} + O(h^2) \right]$$

Contrast the ES of TC model and $\mathbb{Z}_2$ GH model

- When $\theta = \frac{\pi}{2}$ ($\theta \neq \frac{\pi}{2}$), ES are the same (different).

$$\rho_{GH} = K \, \rho_{TC} \, K^\dagger$$



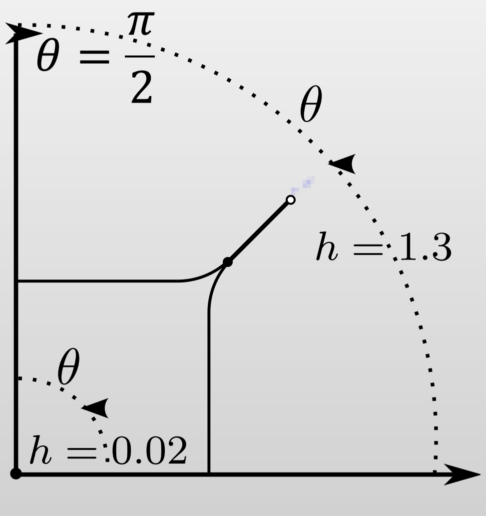
- EM duality for TC

$$\rho_{TC}(h_z, h_x) = U_{E,TC} \, \rho_{TC}(h_x, h_z) \, U_{E,TC}^\dagger$$

- From $\rho_{GH} = \mathcal{N}(\rho_{TC})$, we have

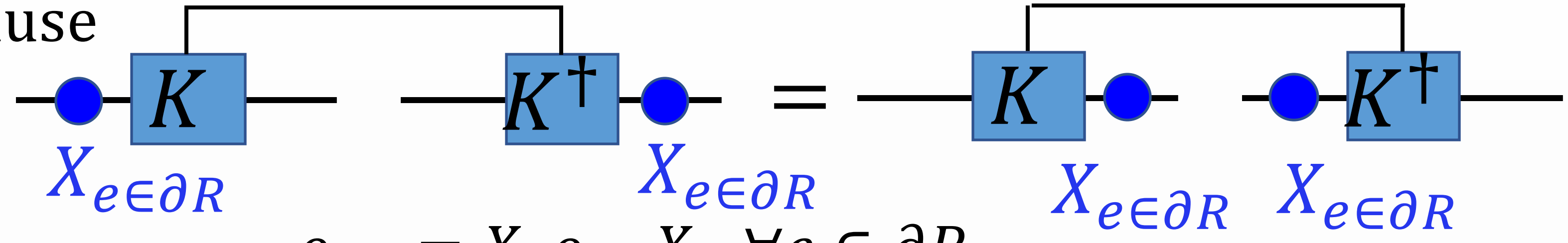
$$\rho_{GH}(h_z, h_x) = K \, U_{E,TC} \, K^\dagger \, \rho_{GH}(h_x, h_z) \, K \, U_{E,TC}^\dagger \, K^\dagger$$

$$= U_{E,TC} \, \rho_{GH}(h_x, h_z) \, U_{E,TC}^\dagger \neq U_{E,GH} \rho_{GH}(h_x, h_z) U_{E,GH}^\dagger$$

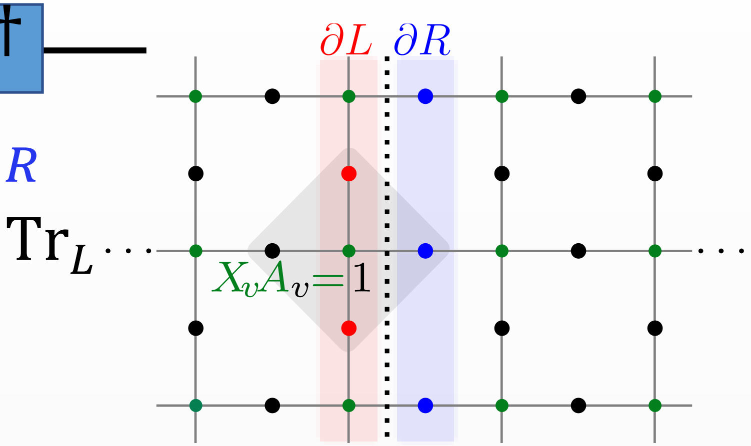


ρ_{GH} is block diagonal

- Because



$$\rho_{GH} = X_e \rho_{GH} X_e, \forall e \in \partial R$$



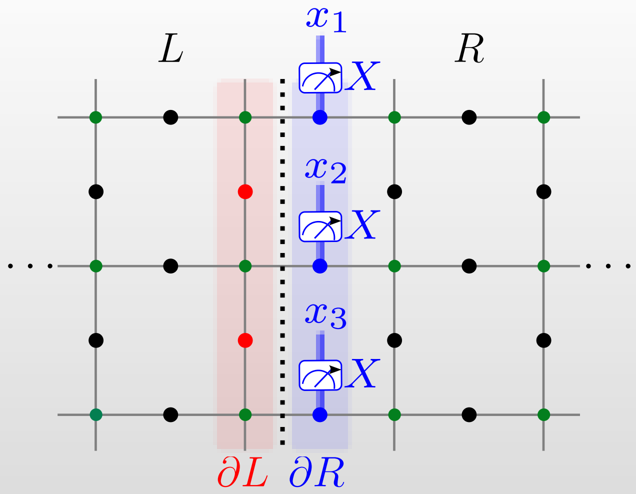
- ρ_{GH} is block diagonal:

$$\rho_{GH} = \bigoplus_{\boldsymbol{x}} p_{\boldsymbol{x}}\, \rho_{GH,\boldsymbol{x}},$$

Notation: $|\boldsymbol{x}\rangle = \prod_{e\in\partial R} |x_e\rangle$, $x_e = \pm 1$: eigenvalue of X_e.

- Ensemble of states from measurement: $\{|\Psi_{GH,\boldsymbol{x}}\rangle\}$
- Probability distribution:

$$p_{GH,\boldsymbol{x}} = \langle \Psi_{GH,\boldsymbol{x}} | \Psi_{GH,\boldsymbol{x}} \rangle,$$



- Subblock RDM

$$\rho_{GH,\boldsymbol{x}} = \frac{\mathrm{Tr}_L |\Psi_{GH,\boldsymbol{x}}\rangle\langle\Psi_{GH,\boldsymbol{x}}|}{p_{GH,\boldsymbol{x}}}$$

P. Buividovich, et. al, physics letter B 670, 141 (2008)
W. Donnelly, PRD, 85,085004,(2012);
H. Casini, et. al, PRD, 89,085012 (2014)

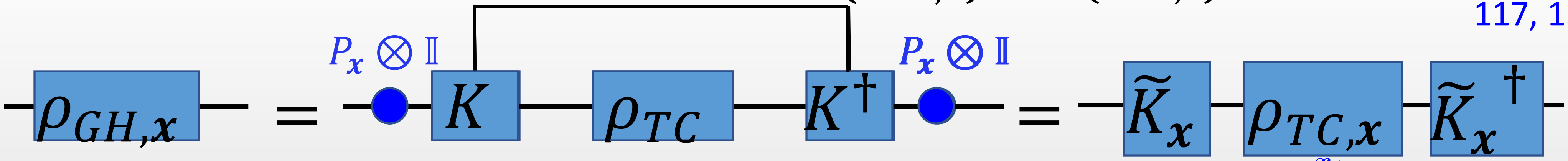
Study distillable EE via quantum channel

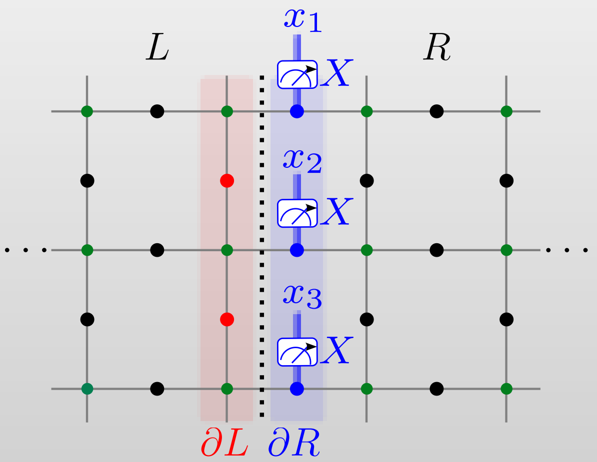
- Entanglement entropy (EE) of $\rho_{GH} = \oplus_x p_x \rho_{GH,x}$:

$$S = \mathrm{Tr}_L \rho_{GH} \log \rho_{GH} = -\sum_x p_x \log p_x + S_D$$

- S_D is the *distillable EE*: $S_D = \sum_x p_x S(\rho_{GH,x})$
- S_D: entanglement that can be accessed by gauge invariant operations.

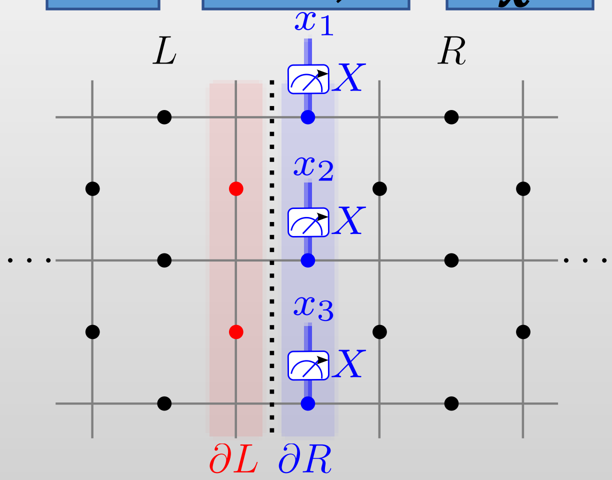
- Subblock EE of $\mathbb{Z}_2$ GH model from TC: $S(\rho_{GH,x}) = S(\rho_{TC,x})$, because





$\mathbb{Z}_2$ GH

TC:



P. Buividovich, et. al, physics letter B 670, 141 (2008)

W. Donnelly, PRD, 85,085004,(2012)

H. Casini, et. al, PRD, 89,085012 (2014)

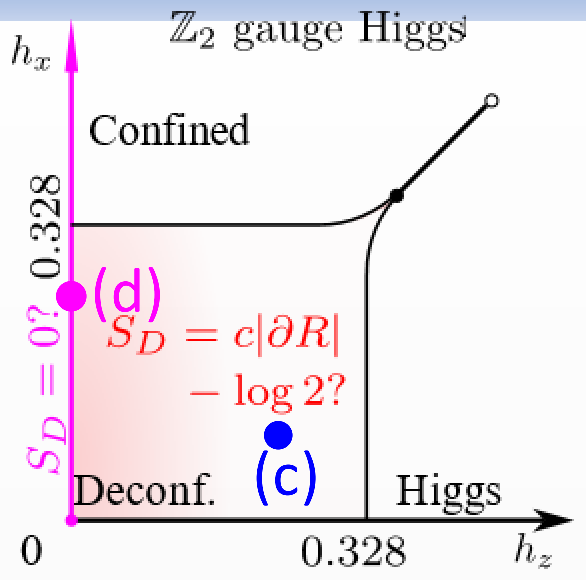
Karel Van Acoleyen, PRL 117, 131602 (2016)

Area law of the distillable EE

- Conjectures:

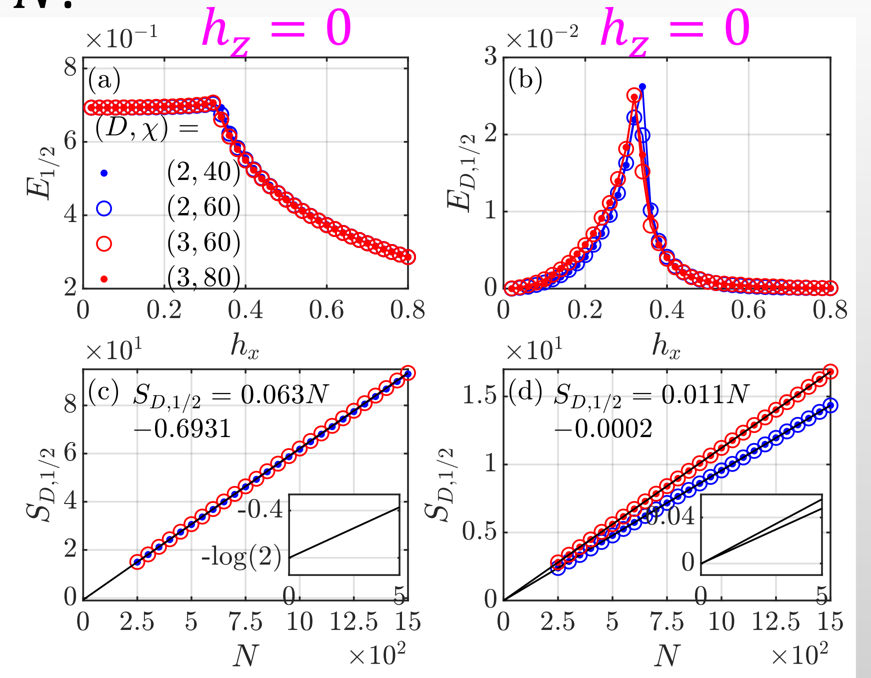
Karel Van Acoleyen, PRL 117, 131602 (2016)

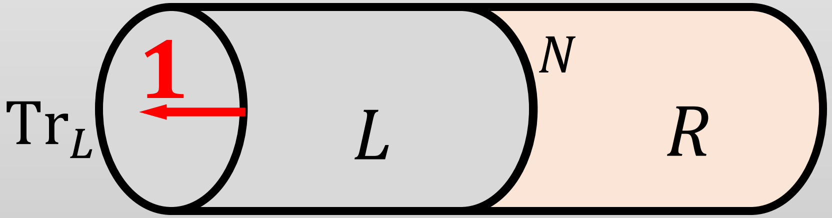
1. For pure gauge along h_x axis $(h_z = 0)$, $S_D = 0$
2. In the deconfined phase with matter field, $S_D = c|\partial L| - \log 2$.



- Efficient tensor network method to calculate S_D for large N.

1. $S_D = c|\partial R| - 0$ for pure $\mathbb{Z}_2$ gauge theory
2. $S_D = c|\partial R| - \log 2$ for $\mathbb{Z}_2$ gauge theory with matter field in the deconf. phase.

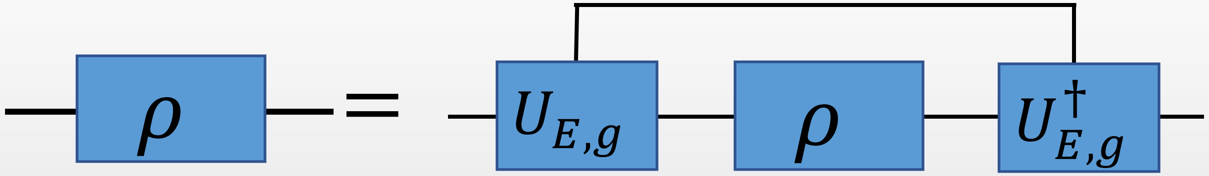




Summary and outlook

- Using the quantum channel we can

1. Extract entanglement of $\mathbb{Z}_2$ GH model from TC
2. Analysis how symmetries (EM duality) apply on RDM of $\mathbb{Z}_2$ GH model
3. Entanglement distillation of $\mathbb{Z}_2$ GH model from TC



- Outlook:

1. A practical approach to study entanglement of other gauge theories
2. Study entanglement of non-trivial SPT (SET) from trivial one