A Topological Classification of Time Reversal Symmetric Frustrated Systems and Metamaterials

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Tuebingen, 7th December 2023

Goal and results

$\lvert\nu\rvert$	Dimension d						
	1	2	3	4		5	
			$m \geq 2$	$m = 2$	$m \geq 3$	$m = 2$	$m \geq 3$
0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
1	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
2	0	0	0		0		$\mathbb{Z}$
≥ 3	0	0	0		0		0

AIII

$\lvert\nu\rvert$	d			
	1	2	3	
			$m = 1$	$m \geq 2$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\star$
1	0	0		$\star$
≥ 2	0	0		0

AIII/BDI

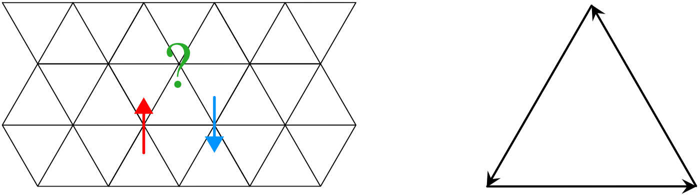
$\lvert\nu\rvert$	d		
	1	2	3
0	$\mathbb{Z}$	$\mathbb{Z}$	$\star$
≥ 2	0	0	0

AIII/CII

Introduction

Motivation

- ``Frustration'' describes the situation where spins in a spin model cannot find an orientation to minimise the interaction energies with their neighbouring spins simultaneously[1]



(a) (b)

Figure: Antialignment of each spin in Heisenberg antiferromagnet (HAF) with nn interactions on a triangular lattice (a) is impossible. A cluster of three spins (b) forms a unique structure.

[1] H.T. Diep. *Frustrated Spin Systems*. World Scientific, 2004, p. 2.

Motivation

- Ground states (GSs) of HAFs are determined by satisfying certain constraints in each cluster, e.g. zero total spin[2]
- Example Hamiltonian ($J > 0$)

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = \frac{J}{2} \sum_{\alpha} |\mathbf{L}_\alpha|^2 + c$$

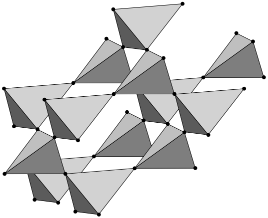
with

$$\mathbf{L}_\alpha := \sum_{i \in \alpha} \mathbf{S}_i$$

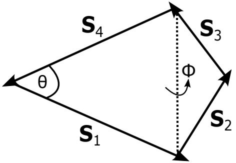
[2] Roderich Moessner and Arthur Ramirez. ``Geometrical Frustration''. In: *Physics Today* 59 (Feb. 2006).

Maxwell counting argument

- The hallmark of frustration is a large **accidental** GS degeneracy
- Estimate[3] ν := #GS DOFs per unit cell $= N - M$ with N := #Total spin DOFs per unit cell and M := #Linearly independent GS constraints per unit cell



(a)

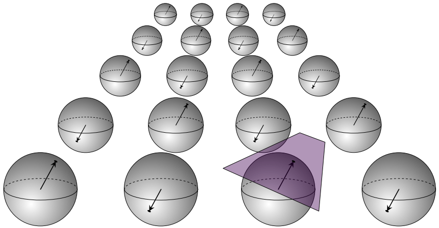


(b)

Figure: GSs of the pyrochlore (a) HAF are characterised by a vanishing total spin (b) in each tetrahedron and parameterised by $\nu = 2$ DOFs θ and ϕ.

[3] R. Moessner and J. T. Chalker. ``Properties of a Classical Spin Liquid: The Heisenberg Pyrochlore Antiferromagnet''. In: *Phys. Rev. Lett.* 80 (13 Mar. 1998).

Linearised degrees of freedom and constraints



- Néel ordered state is one of many GSs of the $J_1 - J_2$ HAF on a square lattice[4]
- Expand around chosen GS $\rightarrow$ linearised DOFs come from plane (purple) perpendicular to fixed spin axis (black dot), i.e. tangent space to sphere (grey) S^2

[4] Krishanu Roychowdhury and Michael J. Lawler. ``Classification of magnetic frustration and metamaterials from topology''. In: *Phys. Rev. B* 98 (9 Sept. 2018).

Classification outline

- Classify topology of zero modes in frustrated systems as function of GS degeneracy homotopically[5]
- Origin of frustration: accidental degeneracy of zero modes $\rightarrow$ topological invariants
- Methods similar to derivation of Bott-Kitaev table
- E.g. flattening of singular values instead of spectral flattening of Hamiltonians

[5] Roychowdhury and Lawler, ``Classification of magnetic frustration and metamaterials from topology''.

Physical framework

The spaces of linearised DOFs and constraints

- $\mathbb{Z}^d$ = Underlying lattice, and associate to each lattice position a $\mathbb{C}^N$ = Unit cell of linearised DOFs of a spin wave in a frustrated system
- Linearised degrees of freedom live in

$$\mathcal{H}_d^N := \ell^2\left(\mathbb{Z}^d, \mathbb{C}^N\right)$$
$$= \left\{ \varphi : \mathbb{Z}^d \to \mathbb{C}^N \;\middle|\; \sum_{i=1}^{N} \sum_{\mathbf{x}\in\mathbb{Z}^d} |\varphi_i(\mathbf{x})|^2 < \infty \right\}$$

- Models large GS degeneracy
- The GS constraints live in $\mathcal{H}_d^M$

Rigidity matrices

- **Rigidity operator**

$$R\colon \mathcal{D}(R) \subseteq \mathcal{H}_d^N \to \mathcal{H}_d^M$$

$$\text{Linearised DOFs} \to \text{Constraints},$$

- Corresponding linearised Hamiltonian $H = R^\dagger R$ governing spin waves dynamics
- $\ker H = \ker R$ contains the zero modes
- Topological classification of translation invariant rigidity operators $\to$ explore new varieties of frustration in which zero modes are demanded from topology[6]
- Classify the topology of zero modes in frustrated systems

[6] Roychowdhury and Lawler, ``Classification of magnetic frustration and metamaterials from topology''.

Rigidity matrices

- Fourier transform $F\colon \mathcal{H}_d^N \to \mathcal{K}_d^N := L^2\left(T^d, \mathbb{C}^N\right)$ turns R into multiplication operator $FRF^\dagger$
- Multiplication by the continuous based **rigidity matrix map** $r\colon T^d \to \mathbb{C}^{M\times N}$ on the Brillouin zone $T^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$
- $\operatorname{rank} r \equiv \min(N, M)$ implements the linear independence assumption of GS constraints
- Gap condition: number of nonzero singular values is rank of the matrix and the only way a zero mode can be introduced and a gap closed is to reduce this rank
- **Maxwell counting indices** in terms of rigidity matrices, $\nu = \mathrm{nullity}\, r - \mathrm{nullity}\, r^T$ (rank-nullity theorem)

Imposing time reversal symmetry (TRS)

- For time reversal symmetric frustrated systems we have $RT_1 = T_2 R$ with $T_i^2 = \pm \mathrm{Id}$ (both real or quaternionic structures)
- Its rigidity matrix map becomes $\mathbb{Z}_2$-equivariant, i.e.

Label	TRS	$\mathbb{Z}_2$-equivariance condition on r
$AIII$	no	trivial $\mathbb{Z}_2$-equivariance
$AIII/BDI$	yes, $T_i^2 = +\mathrm{Id}$	$r(-\mathbf{k}) = \overline{r(\mathbf{k})}$
$AIII/CII$	yes, $T_i^2 = -\mathrm{Id}$	$r(-\mathbf{k}) = \left(I_{M/2} \otimes \sigma_2\right) \overline{r(\mathbf{k})} \left(I_{N/2} \otimes \sigma_2\right)$

Example: Triangular lattice HAF

- Triangular lattice $\Lambda := \mathbf{a}_1\mathbb{Z} \oplus \mathbf{a}_2\mathbb{Z}$ with $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (1/2, \sqrt{3}/2)$
- The Hamiltonian ($J > 0$)

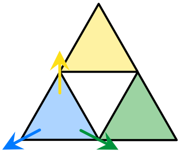
$$H = J \sum_{\mathbf{x}\in\Lambda} \left(S_{\mathbf{x}} S_{\mathbf{x}+\mathbf{a}_1} + S_{\mathbf{x}} S_{\mathbf{x}+\mathbf{a}_2} + S_{\mathbf{x}+\mathbf{a}_1} S_{\mathbf{x}+\mathbf{a}_2} \right)$$
$$= \frac{J}{2} \sum_{\mathbf{x}\in\Lambda} \left(S_{\mathbf{x}} + S_{\mathbf{x}+\mathbf{a}_1} + S_{\mathbf{x}+\mathbf{a}_2} \right)^2 + \mathrm{const.}$$

- GSs are defined by $L_{\mathbf{x}} = S_{\mathbf{x}} + S_{\mathbf{x}+\mathbf{a}_1} + S_{\mathbf{x}+\mathbf{a}_2} = 0$ for all $\mathbf{x} \in \Lambda$

Example: Triangular lattice HAF

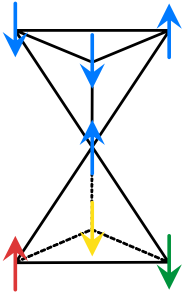
- Linearize the spins around spin axis in ground state with $(q,p) \mapsto \left(\cos(q)\sqrt{1-p^2}, \sin(q)\sqrt{1-p^2}, p\right)$
- Rigidity matrix in position space is defined by $L_{\mathbf{x}} = R\begin{pmatrix} q \\ p \end{pmatrix}$
- Momentum space representation is $\mathbb{Z}_2$-equivariant

$$r(\mathbf{k}) = \begin{pmatrix} \frac{\sqrt{3}}{2}\left(e^{ik_y} - e^{ik_x}\right) & 0 \\ 1 - \frac{1}{2}\left(e^{ik_x} + e^{ik_y}\right) & 0 \\ 0 & 1 + e^{ik_x} + e^{ik_y} \end{pmatrix}$$



- Symmetry class $\mathrm{AIII}/\mathrm{BDI}$ and $\nu = -1$

Example: Pyrochlore HAF



- Similar analysis leads to

$$r(\mathbf{k}) = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -e^{ik_x} & e^{ik_y} & -e^{ik_z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & e^{ik_x} & e^{ik_y} & e^{ik_z} \end{pmatrix}$$

- Symmetry class $\mathrm{AIII}/\mathrm{BDI}$ and described by two individual $\nu = 2$ systems

The mapping space of rigidity matrices

- Define action of $\mathbb{Z}_2 = \{I, g\}$ on mapping space $C\left(T^d, [\mathbf{0}]; \mathbb{C}^{M\times N}, r_0\right)$ by conjugation $(g, r) \mapsto \left(\mathbf{k} \mapsto gr\left(g^{-1}\mathbf{k}\right)\right)$
- $\mathbb{Z}_2$-fixed points $C\left(T^d, [\mathbf{0}]; \mathbb{C}^{M\times N}, r_0\right)^{\mathbb{Z}_2}$ are $\mathbb{Z}_2$-equivariant maps
- Subspace R^N_{dM} in which the *singular value flattened* maps are $\mathbb{Z}_2$-equivariant and based too is the mapping space of rigidity matrices
- Based map condition

$$r(\mathbf{0}) = r_0 := \begin{cases} \begin{pmatrix} I_M & 0 \end{pmatrix} & \text{for } M \leq N, \\ \begin{pmatrix} I_N \\ 0 \end{pmatrix} & \text{for } M \geq N, \end{cases}$$

Spectral flattening technique and classifying spaces

- Classification by the topological invariants $R^N_{dM}/\sim$ (modulo $\mathbb{Z}_2$-homotopy) classifying the topology of zero modes in frustrated systems
- There is a strong deformation retract (homotopy equivalence)

$$\tilde{R}^N_{dM} \cong C(T^d, [\mathbf{0}]; V_n(\mathbb{C}^m), E)^{\mathbb{Z}_2}$$

by linearly interpolating from the matrix of singular values to r_0, with $m := \max(M, N)$, $n := \min(M, N)$ and $E := \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$

- **Classifying spaces** are therefore **Stiefel manifolds** (homogeneous spaces)

$$V_n(\mathbb{F}^m) := \left\{ \Lambda \in \mathbb{F}^{m \times n} \,\middle|\, \Lambda^\dagger \Lambda = I_n \right\}$$

for all $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

Topological invariants

- **Strong** (in the presence of disorder breaking translational symmetry) and **weak** topological invariants are contained in

$$\left[\left(T^d, [\mathbf{0}]\right), \left(V_n\left(\mathbb{C}^m\right), E\right)\right]_{\mathbb{Z}_2}$$

- Replacement of T^d by d-sphere $S^d \simeq I^d/\partial I^d$, with $I := [-\pi, \pi]$, gives the strong invariants[7]

$$\left[\left(I^d, \partial I^d\right), \left(V_n\left(\mathbb{C}^m\right), E\right)\right]_{\mathbb{Z}_2} \cong \pi_0\left(\left(\Omega^d V_n\left(\mathbb{C}^m\right)\right)^{\mathbb{Z}_2}\right)$$

- ***d*-fold iterated loop space** of a based space (X, x_0) is

$$\Omega^d X := \left\{f\colon I^d \to X \mid f\left(\partial I^d\right) = \{x_0\}\right\}$$

- In the absence of TRS we obtain as topological invariants

$$\left[\left(I^d, \partial I^d\right), \left(V_n(\mathbb{C}^m), E\right)\right] = \pi_d\left(V_n(\mathbb{C}^m)\right)$$

[7] Krishanu Roychowdhury et al. ``Supersymmetry on the lattice: Geometry, Topology, and Spin Liquids''. In: 2022.

Comparing topological invariants

The time reversal related symmetries of Roychowdhury and Lawler (2018)[8] lead to the topological invariants

$$\left[\left(I^d, \partial I^d\right), \left(V_n(\mathbb{C}^m)^{\mathbb{Z}_2}, E\right)\right] \cong \pi_d\left(V_n(\mathbb{C}^m)^{\mathbb{Z}_2}\right),$$

i.e. the higher homotopy groups of

$$V_n(\mathbb{C}^m)^{\mathbb{Z}_2} \cong \begin{cases} V_n\left(\mathbb{C}^m\right) & \text{no symmetries,} \\ V_n\left(\mathbb{R}^m\right) & \text{for } T_i^2 = +\mathrm{Id}, \\ V_{n/2}\left(\mathbb{H}^{m/2}\right) & \text{for } T_i^2 = -\mathrm{Id}. \end{cases}$$

[8] Roychowdhury and Lawler, ``Classification of magnetic frustration and metamaterials from topology''.

Reformulation with relative homotopy (groups)

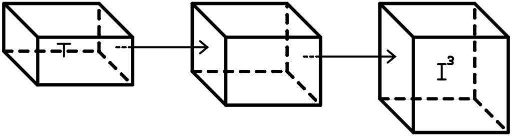
In the presence of TRS we are inspired by R. Kennedy and M.R. Zirnbauer (2015, Lemma 5.13) and find

Lemma 1

For any $\mathbb{Z}_2$-space X ($D, d \geq 0$),

$$\pi_D\left(\left(\Omega^{d+1}X\right)^{\mathbb{Z}_2}\right) \cong \pi_{D+1}\left(\Omega^d X, \left(\Omega^d X\right)^{\mathbb{Z}_2}\right).$$

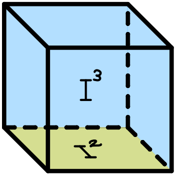
Ingredient: reinterpret relative homotopy groups by $T \cong I^{D+1}$



- Our applications of Lemma 1 are for $D = 0$, $X = V_n(\mathbb{C}^m)$ and base point $E = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}$

The homotopy sequence of a pair

For a based pair of spaces (X, A, x_0) the **boundary operator** $\partial\colon \pi_d(X,A) \to \pi_{d-1}(A)$ is defined by $\partial[f] := \left[f|_{I^{d-1}\times\{-\pi\}} \right]$ (a homomorphism for $d \geq 2$)



Restriction from I^3 onto $I^2 \times \{-\pi\} \cong I^2$

Theorem 2 (Dieck: Algebraic Topology, p. 123)

The following sequence is exact ($d \geq 1$).

$$\cdots \xrightarrow{i_*} \pi_d(X) \xrightarrow{j_*} \pi_d(X,A) \xrightarrow{\partial} \pi_{d-1}(A) \xrightarrow{i_*} \pi_{d-1}(X) \xrightarrow{j_*} \cdots \xrightarrow{i_*} \pi_1(X) \xrightarrow{j_*} \pi_1(X,A) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

Here, i_* and j_* are the induced canonical inclusions.

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$[(I^d, \partial I^d), (X, E)]_{\mathbb{Z}_2}$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$\left[\left(I^d, \partial I^d\right), (X, x_0)\right]_{\mathbb{Z}_2}$$
$$\Vert\wr$$
$$\pi_1\left(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}\right)$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$
\begin{array}{ccccccccc}
 & & & & [(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2} & & & & \\
 & & & & \wr\| & & & & \\
\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(\Omega^{d-1}) & \xrightarrow{j_*} & \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(\Omega^{d-1}) \\
 & & \wr\| & & & & & & \wr\| \\
 & & \pi_d(X) & & & & & & \pi_{d-1}(X)
\end{array}
$$

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$$
\begin{array}{ccccccccc}
 & & & & [(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2} & & & & \\
 & & & & \wr\| & & & & \\
\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(\Omega^{d-1}) & \xrightarrow{j_*} & \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(\Omega^{d-1}) \\
 & & \wr\| & & \vdots & & & & \wr\| \\
 & & \pi_d(X) & & \vdots & & & & \pi_{d-1}(X)
\end{array}
$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$[(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2}$$

$$\| \wr$$

$$\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^{d-1}) \xrightarrow{j_*} \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^{d-1})$$

$$\pi_1(\Omega^{d-1}) \cong \pi_d(X) \qquad \vdots \qquad \pi_0(\Omega^{d-1}) \cong \pi_{d-1}(X)$$

$$\pi_1\left(\Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_1\left(\Omega^2\right) \xrightarrow{j_*} \pi_1\left(\Omega^2, \Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{\partial} \pi_0\left(\Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_0\left(\Omega^2\right)$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$[(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2}$$

$$\wr\|$$

$$\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^{d-1}) \xrightarrow{j_*} \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^{d-1})$$

$$\pi_1(\Omega^{d-1}) \cong \pi_d(X), \qquad \pi_0(\Omega^{d-1}) \cong \pi_{d-1}(X)$$

$$\vdots$$

$$\pi_1(\Omega^2_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^2) \xrightarrow{j_*} \pi_1(\Omega^2, \Omega^2_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^2_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^2)$$

$$\pi_1(\Omega^2_{\mathbb{Z}_2}) \cong \pi_2(\Omega, \Omega_{\mathbb{Z}_2}), \qquad \pi_1(\Omega^2) \cong \pi_3(X), \qquad \pi_0(\Omega^2) \cong \pi_2(X)$$

$$\pi_1(\Omega, \Omega_{\mathbb{Z}_2}) \xrightarrow{\cong} \pi_0(\Omega^2_{\mathbb{Z}_2})$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$[(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2}$$

$$\wr\|$$

$$\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^{d-1}) \xrightarrow{j_*} \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^{d-1})$$

$$\pi_1(\Omega^{d-1}) \cong \pi_d(X), \qquad \pi_0(\Omega^{d-1}) \cong \pi_{d-1}(X)$$

$$\vdots$$

$$\pi_1\left(\Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_1\left(\Omega^2\right) \xrightarrow{j_*} \pi_1\left(\Omega^2, \Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{\partial} \pi_0\left(\Omega^2_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_0\left(\Omega^2\right)$$

$$\pi_1\left(\Omega^2_{\mathbb{Z}_2}\right) \cong \pi_2\left(\Omega, \Omega_{\mathbb{Z}_2}\right), \qquad \pi_1\left(\Omega^2\right) \cong \pi_3(X), \qquad \pi_0\left(\Omega^2\right) \cong \pi_2(X)$$

$$\pi_1\left(\Omega, \Omega_{\mathbb{Z}_2}\right) \xrightarrow{\cong} \pi_0\left(\Omega^2_{\mathbb{Z}_2}\right)$$

$$\pi_1\left(\Omega_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_1\left(\Omega\right) \xrightarrow{j_*} \pi_1\left(\Omega, \Omega_{\mathbb{Z}_2}\right) \xrightarrow{\partial} \pi_0\left(\Omega_{\mathbb{Z}_2}\right) \xrightarrow{i_*} \pi_0\left(\Omega\right)$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$[(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2}$$

$$\wr\|$$

$$\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^{d-1}) \xrightarrow{j_*} \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^{d-1})$$

$$\pi_1(\Omega^{d-1}) \cong \pi_d(X), \qquad \pi_0(\Omega^{d-1}) \cong \pi_{d-1}(X)$$

$$\vdots$$

$$\pi_1(\Omega^2_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega^2) \xrightarrow{j_*} \pi_1(\Omega^2, \Omega^2_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega^2_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega^2)$$

$$\pi_1(\Omega^2_{\mathbb{Z}_2}) \cong \pi_2(\Omega, \Omega_{\mathbb{Z}_2}), \qquad \pi_1(\Omega^2) \cong \pi_3(X), \qquad \pi_0(\Omega^2) \cong \pi_2(X)$$

$$\pi_1(\Omega, \Omega_{\mathbb{Z}_2}) \xrightarrow{\cong} \pi_0(\Omega^2_{\mathbb{Z}_2})$$

$$\pi_1(\Omega_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_1(\Omega) \xrightarrow{j_*} \pi_1(\Omega, \Omega_{\mathbb{Z}_2}) \xrightarrow{\partial} \pi_0(\Omega_{\mathbb{Z}_2}) \xrightarrow{i_*} \pi_0(\Omega)$$

$$\pi_1(\Omega_{\mathbb{Z}_2}) \cong \pi_2(X, X^{\mathbb{Z}_2}), \qquad \pi_1(\Omega) \cong \pi_2(X), \qquad \pi_0(\Omega) \cong \pi_1(X)$$

$$\pi_1(X, X^{\mathbb{Z}_2}) \xrightarrow{\cong} \pi_0(\Omega_{\mathbb{Z}_2})$$

An Algorithm ($X = V_n(\mathbb{C}^m)$, $\Omega^d_{(\mathbb{Z}_2)} \equiv (\Omega^d X)^{(\mathbb{Z}_2)}$)

$$
\begin{array}{ccccccccc}
 & & & & [(I^d, \partial I^d), (X, x_0)]_{\mathbb{Z}_2} & & & & \\
 & & & & \wr\| & & & & \\
\pi_1(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(\Omega^{d-1}) & \xrightarrow{j_*} & \pi_1(\Omega^{d-1}, \Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(\Omega^{d-1}_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(\Omega^{d-1}) \\
 & & \wr\| & & \vdots & & & & \wr\| \\
 & & \pi_d(X) & & & & & & \pi_{d-1}(X) \\
\pi_1(\Omega^2_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(\Omega^2) & \xrightarrow{j_*} & \pi_1(\Omega^2, \Omega^2_{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(\Omega^2_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(\Omega^2) \\
\wr\| & & \wr\| & & & \cong\nearrow & & & \wr\| \\
\pi_2(\Omega, \Omega_{\mathbb{Z}_2}) & & \pi_3(X) & & & & & & \pi_2(X) \\
\pi_1(\Omega_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(\Omega) & \xrightarrow{j_*} & \pi_1(\Omega, \Omega_{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(\Omega_{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(\Omega) \\
\wr\| & & \wr\| & & & \cong\nearrow & & & \wr\| \\
\pi_2(X, X^{\mathbb{Z}_2}) & & \pi_2(X) & & & & & & \pi_1(X) \\
\pi_1(X^{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_1(X) & \xrightarrow{j_*} & \pi_1(X, X^{\mathbb{Z}_2}) & \xrightarrow{\partial} & \pi_0(X^{\mathbb{Z}_2}) & \xrightarrow{i_*} & \pi_0(X)
\end{array}
$$

In the absence of TRS ($|\nu| = m - n$) AIII

Displayed are $\pi_d\left(V_{m-|\nu|}\left(\mathbb{C}^m\right)\right)$[9]

$\lvert\nu\rvert$	Dimension d						
	1	2	3	4		5	
			$m \geq 2$	$m = 2$	$m \geq 3$	$m = 2$	$m \geq 3$
0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
1	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
2	0	0	0		0		$\mathbb{Z}$
≥ 3	0	0	0		0		0

- The case $\pi_d(\mathrm{U}(1)) = \pi_d\left(S^1\right) = 0$ for all $d \geq 2$ is not explicitly contained in this table from $d \geq 3$

[9] Maurice E. Gilmore. ``Complex Stiefel Manifolds, some homotopy groups and vector fields''. In: *Bulletin of the American Mathematical Society* 73.5 (1967); Mamoru Mimura and Hiroshi Toda. ``Homotopy groups of symplectic groups''. In: *Journal of Mathematics of Kyoto University* 3.2 (1963); A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002, p. 339.

In the presence of TRS

Displayed are $\left[(I^d, \partial I^d), \left(V_{m-|\nu|}(\mathbb{C}^m), E\right)\right]_{\mathbb{Z}_2}$

$|\nu|$	d = 1	2	3, $m=1$	3, $m\geq 2$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\star$
1	0	0		$\star$
≥ 2	0	0		0

AIII/BDI

$|\nu|$	d = 1	2	3
0	$\mathbb{Z}$	$\mathbb{Z}$	$\star$
≥ 2	0	0	0

AIII/CII

- $\star =$ yet to be evaluated strong topological invariants which indicate emergence of unstable regime
- There is trivial regime $\left[(I^d, \partial I^d), \left(V_{m-|\nu|}(\mathbb{C}^m), E\right)\right]_{\mathbb{Z}_2} = 0$ for $|\nu| \geq \lceil d/2 \rceil$ and the unstable regime for $|\nu| < \lceil d/2 \rceil$
- In AIII/BDI we always have $\left[(I^d, \partial I^d), (S^1, 1)\right]_{\mathbb{Z}_2} \cong \mathbb{Z}$ for all $d \geq 1$

Example: The $J_1 - J_2$ HAF on a square lattice

→ Class $\mathrm{AIII}/\mathrm{BDI}$

→ We have the integers $d = 2$ and $N = 2$

→ In Néel state: $M = 6$, at critical point (highly frustrated): $M = 2$, in frustrated state: $M = 4$[10]

→ Classifying spaces are $V_2(\mathbb{C}^6)$, $\mathrm{U}(2)$ and $V_2(\mathbb{C}^4)$, respectively

→ 0, $\mathbb{Z}$ and 0, respectively

[10] Roychowdhury and Lawler, ``Classification of magnetic frustration and metamaterials from topology''.

Summary & Outlook

- → Homotopical classification of zero modes in frustrated systems in presence or absence of canonical TRS
- → Describe all zero modes in the framework of rigidity operators R ($\ker R$ = Space of zero modes)
- → $\exists$*Nonisostatic* systems ($\nu \neq 0$) with nontrivial topological invariants; beyond original Kane and Lubensky[11] *isostatic* class $\nu = 0$
- → Novel topological invariants in presence of canonical TRS compared to Roychowdhury and Lawler (2018)
- → Further symmetry classes, e.g. $\mathrm{AIII}/\mathrm{CI}$ ($r(-\mathbf{k}) = r(\mathbf{k})^T$) and $\mathrm{AIII}/\mathrm{DIII}$ ($r(-\mathbf{k}) = -r(\mathbf{k})^T$)[12]

[11] C. L. Kane and T. C. Lubensky. ``Topological boundary modes in isostatic lattices''. In: *Nature Physics* 10.1 (Dec. 2013).

[12] Roychowdhury et al., ``Supersymmetry on the lattice: Geometry, Topology, and Spin Liquids''.

Thank you

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